A Temperley—Lieb Category for $\mathfrak{sl}_2$-Crystals

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New Developments in Tensor Categories
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I. $\mathfrak{sl}_2$ - Crystals (Kashiwara '90-'91, for general $\mathfrak{g}$)

- $\mathbb{K}$ field, char $\mathbb{K} = 0$.
  Let $A := \mathbb{K}[q]_{(p(0)=0)} \subset \mathbb{K}(q)$.
- Given $V \in U_q(\mathfrak{sl}_2)$-Mod
  $\rightsquigarrow$ a "nice" A-submodule $\mathcal{L} \subset V$ stable under
  Kashiwara's operators $\tilde{E}, \tilde{F}$

renormalization of the Chevalley operators E, F

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  $\rightsquigarrow$ a basis $\mathcal{B}$ of $\mathcal{L}/q\mathcal{L}$ satisfying
  1. $\tilde{E}\mathcal{B} \subset \mathcal{B} \cup \{0\}$, $\tilde{F}\mathcal{B} \subset \mathcal{B} \cup \{0\}$
  2. $\forall\, b, b' \in \mathcal{B}$, $b = \tilde{E}b' \iff b' = \tilde{F}b$

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  called a Crystal Basis for V.

- Crystal Basis are simple combinatorial models of representations.

<u>Example</u>: $V = L(3) \oplus L(2)$

$$\rightsquigarrow \quad \mathcal{B} = \bullet \xrightarrow{\tilde{F}} \bullet \xrightarrow{\tilde{F}} \bullet \xrightarrow{\tilde{F}} \bullet \;\sqcup\; \bullet \xrightarrow{\tilde{F}} \bullet \xrightarrow{\tilde{F}} \bullet$$

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- A crystal morphism $\mathcal{B} \xrightarrow{\varphi} \mathcal{B}'$ is a function $\mathcal{B} \amalg \{0\} \xrightarrow{\varphi} \mathcal{B}' \amalg \{0\}$ such that $\varphi \tilde{E} = \tilde{E} \varphi$, $\varphi \tilde{F} = \tilde{F} \varphi$.

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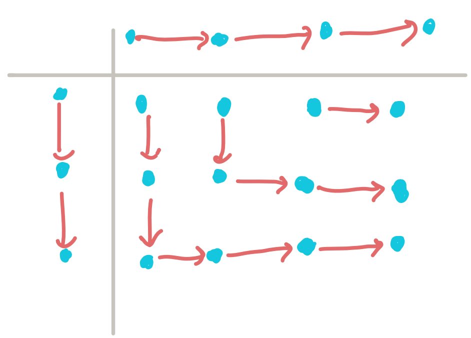
- $\otimes$ of crystals has easy combinatorial rule compatible with $\otimes$ of representations; e.g.



$\rightsquigarrow$ $B(2) \otimes B(3) \cong B(1) \oplus B(3) \oplus B(5)$

$\Rightarrow$ $\underline{\mathfrak{sl}_2\text{-Crys}}$: $\mathbb{K}$-linear Category

- objects: $\mathfrak{sl}_2$-Crystals
- morphisms: $\mathbb{K}$-linear combinations of Crystal morphisms
- monoidal via $\otimes$
- additive (semisimple) via $\amalg$

- has a coboundary commutor (Heriques, Kamnitzer '14)

$$\sigma_{B,B'} : B \otimes B' \longrightarrow B' \otimes B$$

Hard to describe explicitly

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Note: $\underline{\mathfrak{sl}_2\text{-Crys}}$ is not rigid and does not admit a braiding.

good candidate for counterexamples, e.g. Etingof-Penneys '25, Halbig-Zorman '23

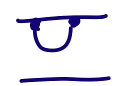
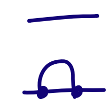
II. The Temperley–Lieb Category (Rumer–Teller–Weyl '32, Jones '83, Wenzl '87, ...)

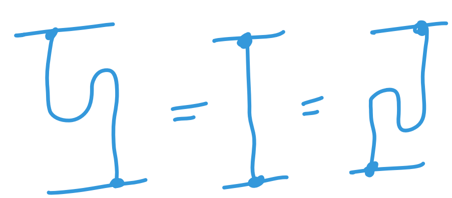
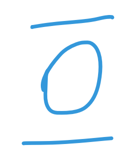
- $[n]_q := \dfrac{q^n - q^{-n}}{q - q^{-1}}$ e.g. $[2]_q = q + q^{-1}$

- $\mathcal{TL}_q$: Strict $\mathbb{K}[q,q^{-1}]$-linear monoidal Category

  Objects : $\mathbb{N} = \{\underline{0}, \underline{1}, \underline{2}, \dots\}$,

  $\mathbb{1} = \underline{0}$, $\underline{n} \otimes \underline{m} = \underline{n+m}$

  morphisms : generated (under $\otimes$, $\circ$) by  $\in \mathrm{Hom}(\underline{0},\underline{2})$,  $\in \mathrm{Hom}(\underline{2},\underline{0})$

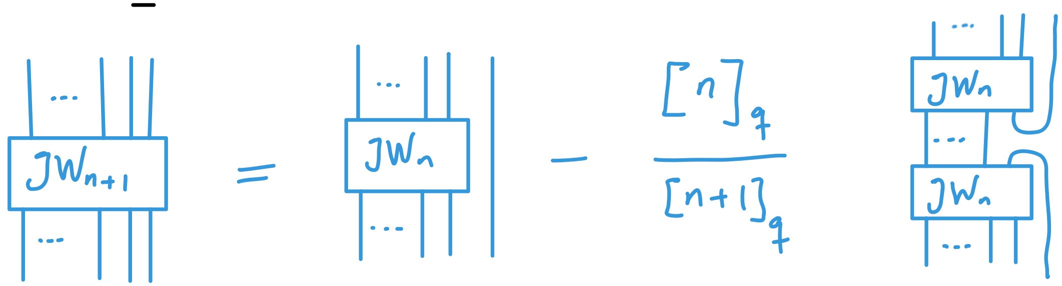
  with relations  ,  $= -[2]_q \, \mathrm{id}_{\underline{0}}$

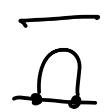
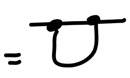
- There is a monoidal Equivalence $\mathcal{TL}_q \simeq \mathrm{Fund}(U_q(\mathfrak{sl}_2))$

$\xrightarrow{\text{Cauchy Completion}} (\mathcal{TL}_q)^c \simeq U_q(\mathfrak{sl}_2)\text{-Mod}$

- Simples: $(\underline{n}, JW_n) \longleftrightarrow L_n$

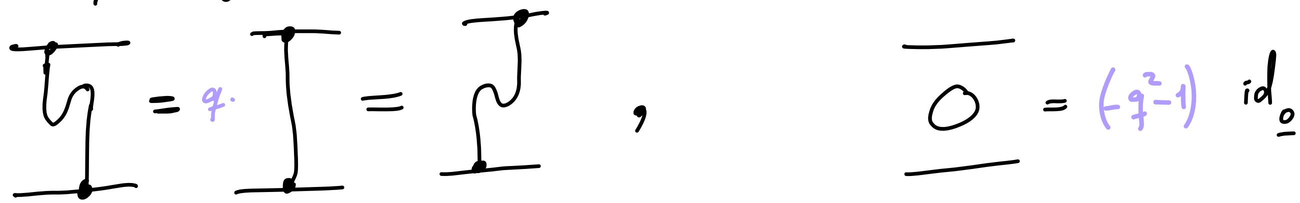
where $JW_1 = id_{\underline{1}}$ and



- Rigid via $\underline{n}^* = \underline{n}$, $ev_{\underline{1}} =$  , $coev_{\underline{1}} =$  and Braided

III. Crystallizing $\mathcal{TL}_q$ (M.A., M. Stroiński '25)

- Define $\widetilde{\mathcal{TL}}_q$ just like $\mathcal{TL}_q$ but modify the relations to



- $\widetilde{\mathcal{TL}}_q \cong \mathcal{TL}_q$ via $\cap \longmapsto q \cdot \cap$

$\rightsquigarrow$ Define $\mathcal{TL}_0 := \widetilde{\mathcal{TL}}_{q=0}$

III. Crystallizing TL_q (M.A., M. Stroiński '25)

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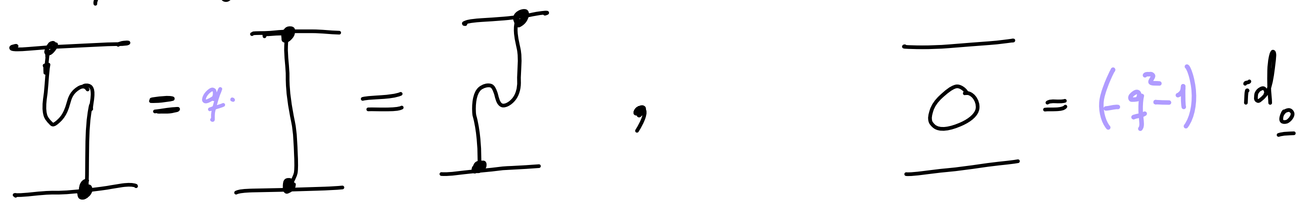


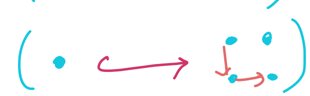
- $\widetilde{TL}_q \cong TL_q$ via $\cap \longmapsto q \cdot \cap$

$\rightsquigarrow$ Define $TL_0 := \widetilde{TL}_{q=0}$

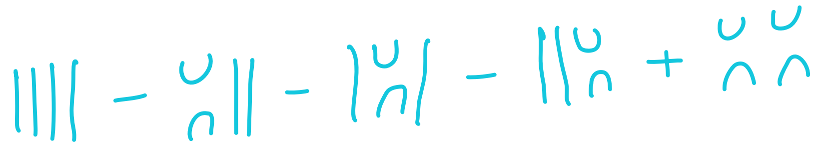
Theorem: • $\underline{CrysTL} := (TL_0)^c \simeq \underline{\mathfrak{sl}_2\text{-Crys}}$ as Coboundary

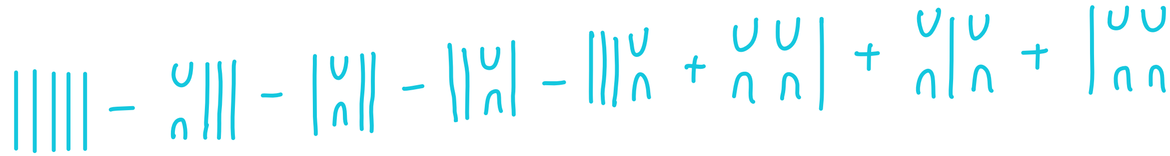
$(\underline{1}, id) \longleftrightarrow B(1) = \bullet \rightarrow \bullet$

$\otimes$ - Categories via $\overline{\cap} \longleftrightarrow$ 

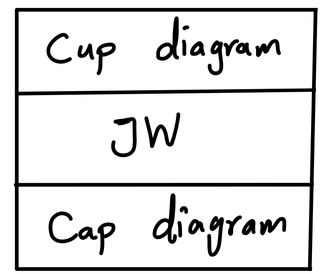
$\underline{\cup} \longleftrightarrow$ 

- JW_n have easy "inclusion – exclusion" closed formula

e.g. $JW_4 =$ 

$JW_5 =$ 

$\rightsquigarrow$ easy semisimple bases :



each simple corresponds to a unique cap/cup diagram

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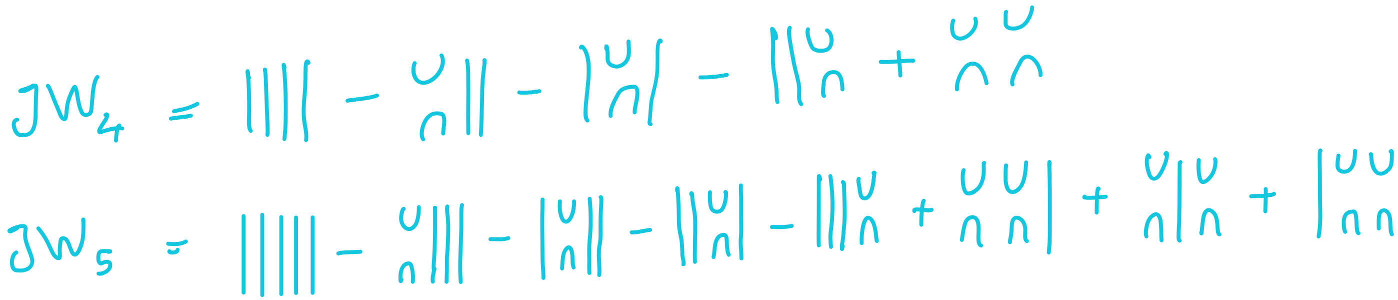
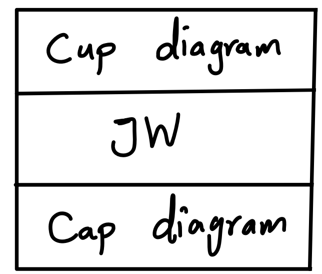


⇝ easy semisimple bases :



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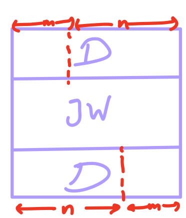
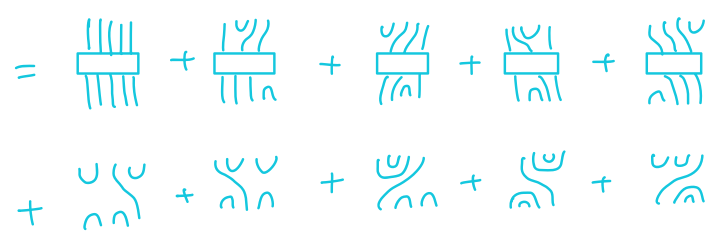
- Coboundary Commutor / Cactus group action have explicit, intuitive diagrammatic formulae :

$$\sigma_{n,m} = \sum_{\text{cap diagrams } D}$$



e.g. $\sigma_{2,3} =$



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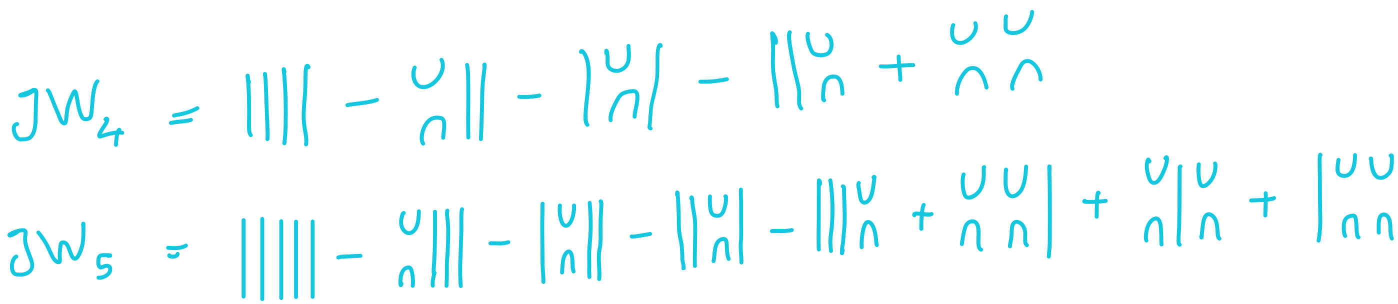

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Cup diagram
JW
Cap diagram

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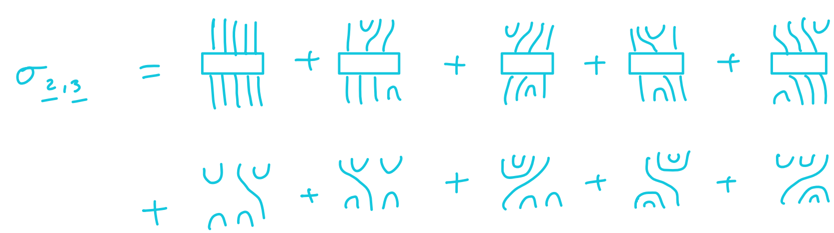

- $\mathcal{TL}_0$ has many more fiber functors than $\mathcal{TL}_{q\neq 0}$.

Thank You for Listening :)

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Full Paper