

THE DRINFELD CENTER OF THE GENERIC TEMPERLEY–LIEB CATEGORY

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OUTLINE

MOTIVATIONS

THE TEMPERLEY–LIEB CATEGORY

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Motivations

BRAIDED TENSOR CATEGORIES

A *braiding* on a tensor category $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, \iota)$ is a natural isomorphism $\sigma_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ satisfying the two *hexagon axioms*:

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$$\begin{array}{ccc}
 & X \otimes (Y \otimes Z) \xrightarrow{\sigma_{X,Y \otimes Z}} (Y \otimes Z) \otimes X & \\
 \alpha_{X,Y,Z} \nearrow & & \searrow \alpha_{Y,Z,X} \\
 (X \otimes Y) \otimes Z & & Y \otimes (Z \otimes X) \\
 \sigma_{X,Y} \otimes \text{Id}_Z \searrow & & \nearrow \text{Id}_Y \otimes \sigma_{X,Z} \\
 & (Y \otimes X) \otimes Z \xrightarrow{\alpha_{Y,X,Z}} Y \otimes (X \otimes Z) &
 \end{array}$$

(Hexagon 1)

And a similar diagram involving α^{-1} instead of α (Hexagon 2)

WHY CARE ABOUT BRAIDED TENSOR CATEGORIES?

- Historical Motivation: Give solutions to the Yang-Baxter Equation, i.e. an operator $R : V \otimes V \rightarrow V \otimes V$ satisfying

$$(R \otimes \text{Id}) \circ (\text{Id} \otimes R) \circ (R \otimes \text{Id}) = (\text{Id} \otimes R) \circ (R \otimes \text{Id}) \circ (\text{Id} \otimes R)$$

- Provide solutions to exactly solvable lattice models in statistical physics
- Gives link and manifold invariants \rightsquigarrow “Quantum Topology”
- Give algebraic framework for 2d Conformal Field Theories, 3d TQFTs, topological quantum computing, . . .

THE DRINFELD CENTER

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DEFINITION

A *half-braiding* on an object X of \mathcal{C} is a natural isomorphism $\varphi : X \otimes - \rightarrow - \otimes X$ satisfying “Hexagon 1”.

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THEOREM (DRINFELD ’86, JOYAL–STREET ’93, MAJID ’91)

Let $\mathcal{Z}(\mathcal{C})$ be the category with

- objects (X, φ) where $X \in \text{Ob}(\mathcal{C})$, and φ is a half-braiding on X ;
- morphisms $f : (X, \varphi) \rightarrow (X', \varphi')$ are morphisms $f : X \rightarrow X'$ in \mathcal{C} intertwining φ and φ' .

Then $\mathcal{Z}(\mathcal{C})$ is a braided monoidal category via

$$(X, \varphi) \otimes (X', \varphi') := (X \otimes X', (\varphi \otimes \text{Id}_{X'}) \circ (\text{Id}_X \otimes \varphi')) \quad \text{and} \quad \sigma_{(X, \varphi), (Y, \psi)} := \varphi_Y.$$

THE DRINFELD CENTER

- Examples:
 1. For a finite-dimensional Hopf algebra A , the center $\mathcal{Z}(\mathbf{Rep}(A)) \simeq \mathbf{Rep}(D(A))$ where $D(A)$ is the *Drinfeld double* of A .
 2. For an algebraic group G , the center $\mathcal{Z}(\mathbf{Rep}(G))$ is equivalent to the category of G -equivariant coherent sheaves on G supported on finitely many points.

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- In general, it is difficult to understand the structure of $\mathcal{Z}(\mathcal{C})$.
- Even for the well-understood categories $\mathbf{Rep}(U_q(\mathfrak{g}))$, the center was unknown!

OUR PROBLEM

- Goal: completely understand the center of the category $\mathbf{Rep}(U_q(\mathfrak{sl}_2))$ of type-I representations (over \mathbb{C}) of quantum \mathfrak{sl}_2 for generic q .

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Remark: this approach was used by Flake–Harman–Laugwitz ('23) successfully to describe the center of Deligne’s category $\underline{\mathbf{Rep}}(S_t)$.

The Temperley–Lieb Category

THE TEMPERLEY–LIEB CATEGORY

Notation: \mathbb{K} is **any** algebraically closed field. $[n]_q = q^{n-1} + q^{n-3} + \cdots + q^{1-n}$.

DEFINITION

Let $\widetilde{\mathbf{TL}}(q; \mathbb{K})$ be the $\mathbb{K}[q, q^{-1}]$ -linear monoidal category defined as follows:

- $\text{Ob}(\widetilde{\mathbf{TL}}(q; \mathbb{K})) = \mathbb{N} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$, with $1_{\widetilde{\mathbf{TL}}(q; \mathbb{K})} = \mathbf{0}$ and $\mathbf{m} \otimes \mathbf{n} = \mathbf{m+n}$;
- The morphisms of $\widetilde{\mathbf{TL}}(q; \mathbb{K})$ are generated by $\text{cup} \in \text{Hom}_{\widetilde{\mathbf{TL}}(q; \mathbb{K})}(\mathbf{0}, \mathbf{2})$ and $\text{cap} \in \text{Hom}_{\widetilde{\mathbf{TL}}(q; \mathbb{K})}(\mathbf{2}, \mathbf{0})$ depicted by string diagrams, modulo the relations

$$\text{cup} = \text{cap} = \text{cup} \quad \text{and} \quad \text{circle} = -[2]_q \text{Id}_{\mathbf{0}}.$$

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THEOREM (JIMBO '86, RUMER–TELLER–WEYL '32 + FOLKLORE)

The Karoubi completion $\mathbf{TL}(q; \mathbb{K})$ of $\widetilde{\mathbf{TL}}(q; \mathbb{K})$ is monoidally equivalent to $\mathbf{Rep}(U_q(\mathfrak{sl}_2))$.

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PROPOSITION (ETINGOF–OSTRIK '03)

Let \mathcal{C} be a tensor category, and X is an object of \mathcal{C} equipped with two morphisms $f : \mathbb{1} \rightarrow X \otimes X$ and $g : X \otimes X \rightarrow \mathbb{1}$ satisfying

$$(\mathrm{Id}_X \otimes g) \circ (f \otimes \mathrm{Id}_X) = \mathrm{Id}_X = (g \otimes \mathrm{Id}_X) \circ (\mathrm{Id}_X \otimes f), \text{ and}$$

$$g \circ f = -[2]_q \mathrm{Id}_{\mathbb{1}}.$$

Then, there is a unique tensor functor $F : \mathbf{TL}(q; \mathbb{K}) \rightarrow \mathcal{C}$ with

$$F(\underline{\mathbb{1}}) = X, \quad F(\mathrm{cup}) = f, \quad F(\mathrm{cap}) = g.$$

STRUCTURE OF $\mathbf{TL}(q; \mathbb{K})$

THEOREM (JONES '83)

Suppose \mathbb{K} is any algebraically closed field \mathbb{K} and $q \in \mathbb{K}^\times$ satisfies $q^2 \neq -1$.

1. There is a unique indecomposable summand T_n of $\underline{\mathbf{n}}$ that does not appear in $\underline{\mathbf{n} - 2}$.
2. $\{T_n\}_{n \in \mathbb{N}}$ is a complete list of indecomposable objects in $\mathbf{TL}(q; \mathbb{K})$.

When q is not a root of unity in \mathbb{K} , we have

3. $\dim_q(T_n) = [n + 1]_q$.
4. $T_m \otimes T_n \cong T_{|m-n|} \oplus T_{|m-n|+2} \oplus \cdots \oplus T_{m+n}$;
5. $\mathbf{TL}(q; \mathbb{K})$ is semisimple.

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– $\otimes T_1$ “fusion graph”: A_∞



ROOT OF UNITY CASE

- For $q \in \mathbb{K}^\times$ a root of unity, $\mathbf{TL}(q; \mathbb{K})$ is not semisimple.
- But we can “semisimplify” it by killing a certain ideal to get a semisimple category $\overline{\mathbf{TL}}(q; \mathbb{K})$.

Notation: q is a primitive $2\kappa^{\text{th}}$ root of unity, and $\mathcal{A}_{\kappa-1} := \overline{\mathbf{TL}}(q; \mathbb{K})$.

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THEOREM (JONES '83)

$\mathcal{A}_{\kappa-1}(\mathbb{K})$ is a fusion category (i.e. semisimple tensor category with finitely many simples) with simple objects T_n with $n = 0, 1, 2, \dots, \kappa - 2$ and $\dim T_n = [n + 1]$. They satisfy the truncated Clebsch-Gordan fusion rule given by

$$T_m \otimes T_n \cong \begin{cases} T_{|m-n|} \oplus T_{|m-n|+2} \oplus \cdots \oplus T_{m+n} & m+n \leq \kappa-2 \\ T_{|m-n|} \oplus T_{|m-n|+2} \oplus \cdots \oplus T_{(2\kappa-4)-(m+n)} & m+n > \kappa-2 \end{cases}.$$

$\mathcal{A}_{\kappa-1}$ fusion graph: ● — ● — ● — ... — ●

BRAIDINGS OF $\mathbf{TL}(q; \mathbb{K})$

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THEOREM (JONES '83)

Generically, $\mathbf{TL}(q; \mathbb{K})$ (and similarly $\mathcal{A}_{\kappa-1}$) admits exactly four braidings given by

$$\begin{array}{c} \diagup \\ \diagdown \end{array} := a \begin{array}{|c|} \hline | \\ \hline \end{array} + a^{-1} \begin{array}{c} \cup \\ \cap \end{array},$$

where $a = \pm q^{\pm 1/2}$.

CENTERS OF $\mathcal{A}_{\kappa-1}$

DEFINITION (MÜGER '03)

X in a braided tensor category \mathcal{C} is *transparent* if $\sigma_{Y,X} \circ \sigma_{X,Y} = \text{Id}_{X \otimes Y}$ for every Y in \mathcal{C} .

CENTERS OF \mathcal{A}_{n-1}

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THEOREM (MÜGER '03, DRINFELD–GELAKI–NIKSHYCH–OSTRIK '10)

Let \mathcal{C} be a braided fusion category whose only transparent simple object is $\mathbb{1}$. Then,

$$D : \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \rightarrow \mathcal{Z}(\mathcal{C})$$

$$X \boxtimes Y \mapsto (X \otimes Y, (\sigma_{X,-} \otimes \text{Id}_Y) \circ (\text{Id}_X \otimes \sigma_{-,Y}^{-1}))$$

is an equivalence of braided tensor categories.

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COROLLARY

$$\mathcal{Z}(\mathcal{A}_{\kappa-1}) \simeq \mathcal{A}_{\kappa-1} \boxtimes \mathcal{A}_{\kappa-1}^{\text{rev}}.$$

A Logical Interlude

ULTRAFILTERS AND ULTRAPRODUCTS

- A *non-principal ultrafilter* on \mathbb{N} is a collection of subsets axiomatizing the notion of “a majority” among \mathbb{N} .
- Fix a non-principal Ultrafilter \mathcal{U} on \mathbb{N} .
- If a property P holds for all integers in some $A \in \mathcal{U}$, we say that P holds for “most” natural numbers.

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DEFINITION

Given a sequence of Rings $\{R_i\}_{i \in \mathbb{N}}$, define their *ultraproduct* $R_{\mathcal{U}}$ as follows: As a set, $R_{\mathcal{U}} = (\prod_{i \in \mathbb{N}} R_i) / \sim_{\mathcal{U}}$, where $(x_1, x_2, x_3, \dots) \sim_{\mathcal{U}} (y_1, y_2, y_3, \dots)$ if and only if $\{i \mid x_i = y_i\} \in \mathcal{U}$. The ring operations are defined component-wise.

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- **Think of $R_{\mathcal{U}}$ as an “ultra-limit” of R_i .**
- Notation: $R_{\mathcal{U}}$ is also denoted $\prod_{\mathcal{U}} R_i$, and the equivalence class of $(x_i)_{i \in \mathbb{N}}$ in $R_{\mathcal{U}}$ is denoted $\text{ulim}_{i \rightarrow \infty} x_i$.

ŁOŚ'S THEOREM

THEOREM (ŁOŚ '55)

A property that can be expressed by “a first-order sentence in the language of rings” holds for $R_{\mathcal{N}}$ if and only if it holds for R_i for “most” $i \in \mathbb{N}$.

- Roughly speaking, “a first-order sentence” is a sentence that uses finitely many quantifications (\exists, \forall) over R and finitely many ring operations (e.g. $+$, \cdot , $-$, etc.).

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- Roughly speaking, “a first-order sentence” is a sentence that uses finitely many quantifications (\exists, \forall) over R and finitely many ring operations (e.g. $+$, \cdot , $-$, etc.).
- For example,

$$\forall x \in R \forall y \in R [x \cdot y = y \cdot x]$$

$$\forall x \in R [(x \neq 0) \rightarrow \exists y \in R [(y \cdot x = 1) \wedge (x \cdot y = 1)]]$$

are first-order sentences in the language of rings, so an ultraproduct of fields is a field.

ULTRAPRODUCTS OF FIELDS

THEOREM (STEINITZ 1910)

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COROLLARY

- $\prod_{\mathcal{U}} \mathbb{C} \cong \mathbb{C}$.
- $\prod_{\mathcal{U}} \overline{\mathbb{F}}_{p_i} \cong \mathbb{C}$, for primes $p_i \rightarrow \infty$.

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PROPOSITION (DELIGNE ’07, CRUMLEY ’10)

Given \mathbb{K}_i -linear **small** tensor categories \mathcal{C}_i , their ultraproduct $\mathcal{C}_{\mathcal{U}}$ is a well-defined small $\mathbb{K}_{\mathcal{U}}$ -linear tensor category.

- Many categorical properties can be expressed in the “first-order language of tensor categories”, e.g. rigidity, semisimplicity, braided/symmetric structure, etc, and thus pass to the ultraproduct by Łoś’s Theorem.

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WARNING!

Finiteness conditions typically cannot be expressed by first-order sentences. For example, the ultraproduct of 1-dimensional vector spaces is \aleph_0 -dimensional.

Main Results

A LITTLE NUMBER THEORY

PROPOSITION (A. '26)

Given $q \in \mathbb{C}$, let \mathbb{K}_i be

- \mathbb{C} if q is transcendental, or
- $\overline{\mathbb{F}}_{p_i}$ if q is algebraic (but not a root of unity), where p_i is a sequence of primes (carefully chosen depending on q) with $p_i \rightarrow \infty$.

There exists primitive $2\kappa_i^{\text{th}}$ roots of unity $q_i \in \mathbb{K}_i$ with $\kappa_i \rightarrow \infty$ and an isomorphism $\mathbb{K}_{\mathcal{U}} \cong \mathbb{C}$ sending $\text{ulim}_{i \rightarrow \infty} q_i$ to q .

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There exists primitive $2\kappa_i^{\text{th}}$ roots of unity $q_i \in \mathbb{K}_i$ with $\kappa_i \rightarrow \infty$ and an isomorphism $\mathbb{K}_{\mathcal{Q}} \cong \mathbb{C}$ sending $\text{ulim}_{i \rightarrow \infty} q_i$ to q .

PROOF SKETCH.

For $q \in \mathbb{Z}$, let $A_i := q^{2^i} + 1$; by Zsigmondy's Theorem, there are primes $p_i \mid A_i$ with $p_i \nmid A_j$ for $j < i$. Observe that q has multiplicative order 2^{i+1} modulo p_i . Generalize to algebraic numbers using a generalized Zsigmondy-type Theorem (Postnikova-Schinzel '68) for algebraic number fields. \square

TL($q; \mathbb{C}$) AS AN ULTRA-LIMIT

- The ultraproduct $\mathcal{A}_{\mathcal{U}}$ of $\mathcal{A}_{\mathcal{U}_i-1}(\mathbb{K}_i)$ is too large!
- Let \mathcal{A}_{∞} be the tensor subcategory of $\mathcal{A}_{\mathcal{U}}$ generated by $X = \text{ulim}_{i \rightarrow \infty} \underline{\mathbf{1}}$.
- Note that $\dim X = \text{ulim}_{i \rightarrow \infty} \dim(\underline{\mathbf{1}}) = \text{ulim}_{i \rightarrow \infty} [2]_{q_i} = [2]_q$.

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COROLLARY

For any $q \in \mathbb{C}^{\times}$ which is not a root of unity, the canonical functor $\mathbf{TL}(q; \mathbb{C}) \rightarrow \mathcal{A}_{\infty}$ is an equivalence of braided tensor categories.

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Problem: Ultraproducts do not commute with Drinfeld Centers in general!

FILTERING $\mathbf{TL}(q)$

- Let $\mathbf{TL}(q)^{\leq r,s}$ and $\mathcal{A}_{\mathcal{X}-1}^{\leq r,s}$ be the full subcategories of $\mathbf{TL}(q)$ and $\mathcal{A}_{\mathcal{X}-1}$ whose objects are $T_{i_1} \oplus T_{i_2} \oplus \cdots \oplus T_{i_a}$ with $a \leq r$ and $i_j \leq s$.
- This induces a filtration on their Drinfeld centers (by looking at the underlying object's filtered component).
- We still have $\mathbf{TL}(q)^{\leq r,s} \simeq \mathcal{A}_{\infty}^{\leq r,s}$ for every $r, s \in \mathbb{N}$.

FILTERING $\mathbf{TL}(q)$

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PROPOSITION

The following functor is an equivalence of categories:

$$\prod_{\mathcal{U}} \mathcal{Z}(\mathcal{A}_{\kappa-1})^{\leq r,s} \rightarrow \mathcal{Z}(\mathcal{A}_{\infty})^{\leq r,s} \simeq \mathcal{Z}(\mathbf{TL}(q; \mathbb{C}))^{\leq r,s}$$

$$\text{ulim}_{i \rightarrow \infty} (X_i, \varphi_i) \mapsto \left(\text{ulim}_{i \rightarrow \infty} X_i, \text{ulim}_{i \rightarrow \infty} \varphi_i \right).$$

Thus, $\mathcal{Z}(\mathbf{TL}(q; \mathbb{C})) \simeq \bigcup_{r,s \in \mathbb{N}} \prod_{\mathcal{U}} (\mathcal{A}_{\kappa-1} \boxtimes A_{\kappa-1}^{\text{rev}})^{\leq r,s}$.

WHAT WE KNOW ABOUT THE CENTER OF $\mathbf{TL}(q)$

- There is a $\mathbb{Z}/2\mathbb{Z}$ -grading on $\mathbf{TL}(q)$, namely $\mathbf{TL}(q) = \mathbf{TL}(q)_0 \oplus \mathbf{TL}(q)_1$ where $\mathbf{TL}(q)_0 = \langle T_{2n} \rangle$ and $\mathbf{TL}(q)_1 = \langle T_{2n+1} \rangle$.

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- This induces a tensor functor

$$G : \mathbf{Rep}(\mathbb{Z}/2\mathbb{Z}) \rightarrow \mathcal{Z}(\mathbf{TL}(q))$$
$$V_{\text{sign}} \mapsto (\mathbf{1}, \xi),$$

where $\xi_{T_n} = (-1)^n \text{Id}_{T_n}$.

WHAT WE KNOW ABOUT THE CENTER OF $\mathbf{TL}(q)$

- There is a $\mathbb{Z}/2\mathbb{Z}$ -grading on $\mathbf{TL}(q)$, namely $\mathbf{TL}(q) = \mathbf{TL}(q)_0 \oplus \mathbf{TL}(q)_1$ where $\mathbf{TL}(q)_0 = \langle T_{2n} \rangle$ and $\mathbf{TL}(q)_1 = \langle T_{2n+1} \rangle$.
- This induces a tensor functor

$$G : \mathbf{Rep}(\mathbb{Z}/2\mathbb{Z}) \rightarrow \mathcal{Z}(\mathbf{TL}(q))$$

$$V_{\text{sign}} \mapsto (\mathbf{1}, \xi),$$

where $\xi_{T_n} = (-1)^n \text{Id}_{T_n}$.

- The braiding σ of $\mathbf{TL}(q)$ also induces a tensor functor

$$G : \mathbf{TL}(q) \boxtimes \mathbf{TL}(q)^{\text{rev}} \rightarrow \mathcal{Z}(\mathbf{TL}(q))$$

$$X \boxtimes Y \mapsto (X \otimes Y, (\sigma_{X,-} \otimes \text{Id}_Y) \circ (\text{Id}_X \otimes \sigma_{-,Y}^{-1})).$$

WHAT WE KNOW ABOUT THE CENTER OF $\mathbf{TL}(q)$

- Thus, we get a tensor functor

$$Z : \mathbf{TL}(q; \mathbb{K}) \boxtimes \mathbf{TL}(q; \mathbb{K})^{\text{rev}} \boxtimes \mathbf{Rep}(\mathbb{Z}/2\mathbb{Z}) \rightarrow \mathcal{Z}(\mathbf{TL}(q; \mathbb{K}))$$

DEFINITION

$$M_{i,j} := Z(T_i \boxtimes T_j \boxtimes \mathbb{1}),$$

$$W_{i,j} := Z(T_i \boxtimes T_j \boxtimes V_{\text{sign}}) \cong M_{i,j} \otimes (\underline{\mathbf{0}}, \xi).$$

WHAT WE KNOW ABOUT THE CENTER OF $\mathbf{TL}(q)$

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LEMMA

$$M_{i,j} \cong \text{ulim}_{k \rightarrow \infty} (L_i \boxtimes L_j) \quad \text{and} \quad W_{i,j} \cong \text{ulim}_{k \rightarrow \infty} (L_{\kappa-2-i} \boxtimes L_{\kappa-2-j}).$$

THE MAIN EQUIVALENCE

THEOREM (A. '26)

For $q \in \mathbb{C}$ not a root of unity, $\mathcal{Z}(\mathbf{TL}(q; \mathbb{C}))$ is semisimple. $\{M_{i,j}, W_{i,j} \mid i, j \in \mathbb{N}\}$ is a complete and irredundant list of its isomorphism classes of simple objects.

COROLLARY (A. '26)

For $q \in \mathbb{C}$ not a root of unity, the tensor functor

$$Z : \mathbf{TL}(q; \mathbb{C}) \boxtimes \mathbf{TL}(q; \mathbb{C})^{\text{rev}} \boxtimes \mathbf{Rep}(\mathbb{Z}/2\mathbb{Z}) \rightarrow \mathcal{Z}(\mathbf{TL}(q; \mathbb{C}))$$

is an equivalence.

BRAIDING OF $\mathcal{Z}(\mathbf{TL}(q; \mathbb{C}))$

- $\mathbf{TL}(q; \mathbb{C}) \boxtimes \mathbf{TL}(q; \mathbb{C}) \boxtimes \mathbf{Rep}(\mathbb{Z}/2\mathbb{Z})$ has a $(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$ -grading
- Consider the bicharacter $\gamma : (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \times (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \rightarrow \mathbb{C}^\times$ given by $\gamma((i, j, k), (i', j', k')) = (-1)^{k(i'+j')}$.

THEOREM (A. '26)

Upon twisting the braiding on $\mathbf{TL}(q; \mathbb{C}) \boxtimes \mathbf{TL}(q; \mathbb{C})^{\text{rev}} \boxtimes \mathbf{Rep}(\mathbb{Z}/2\mathbb{Z})$ by γ , the functor Z becomes an equivalence of braided tensor categories.

Concluding Remarks

EXCEPTIONAL VALUES OF q

- Understanding the center of $\mathbf{TL}(q)$ for q a root of unity remains open!
- Except $q = -1$, because $\mathbf{TL}(q)$ and $\mathbf{TL}(-q)$ are related by a cocycle twist (which commutes with taking Drinfeld centers).
- At $q \rightarrow 0$ (the crystal limit), there is evidence to suggest (but no proof so far) that the center is equivalent to $\mathbf{Rep}(\mathbb{Z}/2\mathbb{Z})$.

OTHER QUANTUM GROUPS

The techniques we used seem promising for computing the centers of $\mathbf{Rep}(U_q(\mathfrak{g}))$ for Lie algebras \mathfrak{g} of higher rank \mathfrak{g} as well!

CONJECTURE

Let P and Q be the weight and root lattices of \mathfrak{g} , respectively. There exists a $(P/Q)^{\oplus 3}$ bicharacter γ such that the canonical functor

$$Z : \left(U_q(\mathfrak{g})\text{-mod} \boxtimes U_q(\mathfrak{g})\text{-mod}^{\text{rev}} \boxtimes \mathbf{Rep}(P/Q) \right)^\gamma \rightarrow \mathcal{Z}(U_q(\mathfrak{g})\text{-mod})$$

is an equivalence of braided tensor categories whenever $q \in \mathbb{C}$ is not a root of unity.

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Thank You :)



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