Fundamental Groups of Wild Spaces

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Abstract

The fundamental group of a topological space is an important algebraic invariant that records information about the different types of loops in the space. It allows us to study topological spaces and classify them to some extent, and it can be used to prove many important results in various mathematical domains. Unfortunately, calculating the fundamental group of a space can be a daunting task. For many nice spaces, the theory of covering spaces provides a general framework for finding their fundamental groups.

In this thesis, we will introduce the classical theory of covering spaces and examine spaces for which the classical theory fails. These are called wild spaces, the canonical example of which is the so-called Hawaiian earring. We will study the Hawaiian earring in detail and describe it as a limit of well-behaved spaces. Applying the theory of covering spaces on the approximating sequence will allow us to compute the fundamental group of the earring itself, which can be utilized to compute fundamental groups of other wild spaces.

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Contents

Chapter 1

Introduction

What is algebraic topology? One of the primary goals of topology is the study of the basic shapes of spaces and the classification of spaces up to homeomorphism. One way to distinguish between topological spaces is via the study of properties that are *invariant* under homeomorphism, e.g. connectedness, compactness, Hausdorffness, etc. For instance, if one space is compact and the other is not, then we can conclude that they are not homeomorphic to one another. Unfortunately, if both spaces were compact, we cannot conclude if they are homeomorphic. Indeed, no combination of such topological properties forms a conclusive test for the homeomorphism of two spaces.

Instead of asking simple yes or no questions such as "is the space compact?", one could hope to capture more of the topological data of a space by assigning to each space a numerical invariant. A famous numerical invariant is the Euler characteristic of a space. A richer family of invariants still is that of algebraic invariants, where we assign to each space an algebraic structure (such as a group or a ring) that carries information about the space, and thereby reducing difficult topological questions to systematic algebraic computations. This is the goal of algebraic topology, which has been incredibly successful in resolving many long standing problems in topology as well as contributing to many other areas. Interestingly, despite their immense usefulness, there is no set of known algebraic invariants capable of answering the question of whether two spaces are homeomorphic in general.

In his seminal 1895 paper "analysis situs", Henri Poincaré introduced the first of these algebraic invariants: *the fundamental group*, which is a group recording information about types of loops in a space. For example, the fundamental group of $\mathbb{R}^2$ is trivial because there is "essentially" one class of loops in the plane (by which we mean that any two loops in the same class can be continuously deformed into one another), whereas the fundamental group of the circle is $\mathbb{Z}$ because loops could wind around the circle twice clockwise (corresponding to $2 \in \mathbb{Z}$), thrice counterclockwise (corresponding to -3), etc. We will rigorously define and compute fundamental groups in Chapter 2. Another way to think about it, is to say that the fundamental group measures the existence of holes in the space. The fundamental group is the first of the *homotopy groups* of a space, the following groups measuring "higher-dimensional holes" in a sense. Together, these homotopy groups almost determine the basic shape of a space. Despite being easy to define, homotopy groups are sometimes very difficult to compute.

For sufficiently nice spaces, the theory of covering spaces provides a framework for understanding the fundamental group of a space. More precisely, it establishes a Galois correspondence between isomorphism classes of coverings of a space on the one hand, and conjugacy classes of subgroups of its fundamental group on the other. This will be examined in details in Chapter 3. The theory fails, however, for spaces that contain at least one point from which it is possible to create arbitrarily small loops that cannot be continuously shrunk to that point. These are called *wild* spaces. In this thesis, we aim to understand methods for computing the fundamental groups of these wild spaces. Although we will not be able to directly apply the results of covering spaces theory to compute fundamental groups of wild spaces, the results and methods developed will still be useful to our analysis.

In Chapter 2, we will define the basic notions of homotopy and the fundamental group of a space. In Chapter 3, we will discuss the classical theory of covering spaces and use it to calculate fundamental groups. Chapter 4 introduces Seifert–van Kampen theorem, an important tool for calculating fundamental groups of "glued" spaces by knowing the fundamental groups of the individual parts. In Chapter 5 we will introduce the notions of an inverse and direct limit, which will help us understand complicated objects via an approximating sequence. Chapter 6 marks the beginning of our examination of wild spaces and introduces the Hawaiian earring, the canonical example of wild spaces, and several related examples. In Chapter 7 we will compute the fundamental group of the Hawaiian earring and examine some of its peculiar algebraic properties. Finally, in Chapter 8 we will examine the fundamental group of the harmonic archipelago, a very wild 2-dimensional space with an even more curious fundamental group.

Chapter 2

The Fundamental Group

In this chapter, we introduce our central object of study: *the fundamental group*. We will also introduce a prototypical example of a fundamental group calculation. For a detailed treatment and more examples, the reader may refer to chapters 6 and 8 of [Cro05].

2.1 Basic Notions of Homotopy

Before we can define fundamental groups as the set "different types" of loops in a space, we need to understand what we mean by two loops being essentially the same. We aim to describe two loops to be similar if one can be continuously "deformed" into the other. Let $I = [0, 1]$.

Definition 2.1. *Given two spaces X and Y, two maps* [1] *$f, g : X \to Y$ are called* homotopic *(denoted $f \sim g$) if there exists a continuous function $H : X \times I \to Y$ such that for every $x \in X$, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. The map H is called a* homotopy *between f and g.*

We think of H as a deformation of f into g happening during the unit time interval I. For example, if $f, g : [-1, 1] \to \mathbb{R}^2$ are the two path given by $f(x) = (x, 1-x^2)$ and $g(x) = (x, x^2 - 1)$, then $H : [-1, 1] \times I \to \mathbb{R}^2$ given by $H(x, t) = (x, (1 - 2t)(1 - x^2))$ is a homotopy between them. Equivalently, one could think of H as a family of continuous functions $\{h_t : X \to Y\}_{t \in I}$, where $h_t(x) = H(x, t)$. The constant loop is sometimes called the *null loop*, and loops homotopic to it are called *nullhomotopic*.

It is easy to verify that $\sim$ is an equivalence relation, and thus one can talk about *homotopy classes* of maps between X and Y, i.e. the quotient of the set of continuous maps by the relation $\sim$; we denote it by $\pi(X, Y)$. By fixing either X or Y and varying the other, we get a map that associates to each topological space a set that we can then equip with a group operation. One important such construction is the fundamental group which will be defined in the next section and will be our object of study for the remainder of this thesis.

[1] Maps between topological spaces are always assumed to be continuous and maps between groups are always assumed to be homomorphisms. Set-theoretic functions between sets with additional structure will be explicitly identified as such.

Definition 2.2. *Two spaces X and Y are called* homotopy equivalent *(denoted $X \simeq Y$) if there are maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim \mathbb{1}_X$ and $f \circ g \sim \mathbb{1}_Y$. In that case, f and g are called* homotopy equivalences*. A space that is homotopic to a point is called* contractible*.*

Homeomorphisms are obviously homotopy equivalences, but the notion of homotopy equivalence is weaker than that of homeomorphism. For example, $\mathbb{R}$ is contractible, but it is not homeomorphic to a point. Algebraic invariants in topology are usually defined up to homotopy equivalence, i.e. they cannot distinguish between homotopy equivalent spaces.

Definition 2.3. *A* retraction *is a map $r : X \to A$, where $A \subseteq X$, such that $r\big|_A = \mathbb{1}_A$. The subspace A is called a* retract *of X.*

A deformation retraction *is a homotopy $F : X \times I \to X$ between the identity and a retraction, i.e. for all $x \in X$ and $a \in A$, $F(x, 0) = x$, $F(x, 1) \in A$ and $F(a, 1) = a$. The subspace A is called a* deformation retract *of X.*

An essentially equivalent definition of a retraction is as a map that has a right-inverse. Notice that a deformation retract of X is homotopy equivalent to X (with the inclusion map being a homotopy equivalence).

2.2 The Group Structure on $\pi_1(X, x_0)$

Let $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle with the subspace topology. We denote by $\pi_1(X, x_0)$ the set $\pi_*(\mathbb{S}^1, X)$ (the subscript $*$ indicates the fact that we treat $\mathbb{S}^1$ and X as based spaces and the maps between them as respecting basepoints), where x_0 is the basepoint of X. In other words, $\pi_1(X, x_0)$ is the set of homotopy classes of loops in X based at x_0. This set forms a group under the operation of concatenation of loops. More precisely, given $f, g : \mathbb{S}^1 \to X$, define their concatenation

$$(f \cdot g)(t) := \begin{cases} f(2t) & \text{if } t \leqslant 1/2 \\ g(2t-1) & \text{if } t \geqslant 1/2 \end{cases},$$

when thinking of $\mathbb{S}^1$ as I with its ends identified. And define the inverse of f by $f^-(t) := f(1-t)$, which is the same loop traversed in the opposite direction. Finally, we extend these definition to homotopy classes by picking an arbitrary representative so that $[f] \cdot [g] = [f \cdot g]$ and $[f]^{-1} = [f^-]$. It is easy to check that these are well-defined and that the group axioms hold. The group $\pi_1(X, x_0)$ under the operation $\cdot$ is called the *fundamental group* of X with basepoint x_0. Since fundamental groups are blind to anything outside the path component of the basepoint, we will always assume that our spaces are path-connected when considering their fundamental groups

Given a continuous map $X \xrightarrow{f} Y$, we have a canonical group homomorphism $f_\# : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ defined by $[\gamma] \mapsto [f \circ \gamma]$. Indeed,

$$\begin{aligned} f_\#([\gamma_1] \cdot [\gamma_2]) &= f_\#([\gamma_1 \cdot \gamma_2]) = [f \circ (\gamma_1 \cdot \gamma_2)] = [(f \circ \gamma_1) \cdot (f \circ \gamma_2)] \\ &= [(f \circ \gamma_1)] \cdot [(f \circ \gamma_2)] = f_\#([\gamma_1]) \cdot (f_\#([\gamma_2]) \end{aligned}$$

Moreover, the induced homomorphisms are compatible with composition of functions, that is $(f \circ g)_\# = f_\# \circ g_\#$. Finally, notice that the identity condition $(\mathbb{1}_X)_\# = \mathbb{1}_{\pi_1(X,x_0)}$ is satisfied. So what we have here is a mapping between topological spaces with identified basepoints and continuous functions between them on the one hand, and groups and group homomorphisms on the other, that satisfies the composition compatibility and identity conditions. Such a map is called a *functor* from the category of pointed spaces to the category of groups.

In order to justify our use of the term "*the* fundamental group of X", we must first show that the algebraic structure of this group does not depend on the choice of the basepoint.

Theorem 2.4. *Given a path-connected space X and any two points $x_0, x_1 \in X$, we have $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

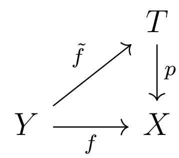
Proof. Since X is path-connected, we have a path $u : I \to X$ with $u(0) = x_0$ and $u(1) = x_1$. Using u, we can construct a map $u_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by setting $u_*([\gamma]) = [u^- \cdot \gamma \cdot u] \in \pi_1(X, x_1)$, where $u^-(t) := u(1-t)$ is the same path traversed in the opposite direction. We can define an inverse map in the same way, $u_*^{-1}([\gamma]) = [u \cdot \gamma \cdot u^-] \in \pi_1(X, x_0)$. Note that these maps are well-defined on homotopy classes since we get a homotopy $u^- \cdot \gamma \cdot u \sim u^- \cdot \gamma' \cdot u$ if $\gamma \sim \gamma'$, by combining the trivial homotopies on u^- and u with the homotopy between γ and γ'. Finally, note that $u_*([\gamma_1] \cdot [\gamma_2]) = [u^- \cdot \gamma_1 \cdot \gamma_2 \cdot u] = [u^- \cdot \gamma_1 \cdot u \cdot u^- \cdot \gamma_2 \cdot u] = u_*([\gamma_1]) \cdot u_*([\gamma_2])$, where the middle equality is because $u^- \cdot u$ is contractible. Thus, u_* is a group isomorphism. □

Notice that the isomorphism u_* depends on the homotopy class of the path u. Therefore, it is still important to mention the basepoint if we want to talk about the fundamental group of X as it relates to fundamental groups of other related spaces (e.g. subspaces of X). If, however, we are only interested in the isomorphism class of the fundamental group of a path-connected space as an abstract algebraic object, we will simply denote it as $\pi_1(X)$. A space X is called *simply connected* if $\pi_1(X)$ is trivial, i.e. if any loop in X can be contracted to its basepoint. The higher homotopy groups are defined similarly with $\mathbb{S}^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\}$ playing the role of $\mathbb{S}^1$, i.e. $\pi_n(X, x_0) = \pi_*(\mathbb{S}^n, X)$. One defines a group structure on $\pi_n(X, x_0)$ in an analogous manner. Like π_1, the mappings π_n are all functors from the category of pointed spaces and continuous functions to the category of groups and group homomorphisms. Denote the image of a continuous map f under this functor by $\pi_n(f)$. A highly non-trivial theorem due to Whitehead tells us that, for a large class of reasonable spaces, a continuous function f such that $\pi_n(f)$ is an isomorphism for every $n \geqslant 1$ is a homotopy equivalence.

2.3 First Example: $\pi_1(\mathbb{S}^1)$

The following example will illustrate the main idea of covering spaces and thus would be helpful in guiding our intuition in the next chapter. First, we need the following definition:

Definition 2.5. *Given a map $p : T \to X$ and a map $f : Y \to X$, a* lifting map *(or simply a* lift*) of f with respect to p is a map $\tilde{f} : Y \to T$ such that $p \circ \tilde{f} = f$, i.e. making the following diagram commute.*



We say that $\tilde{f}$ lifts f with respect to p, and we do not mention p when it is clear from the context.

Theorem 2.6. $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$

Proof. Let $p : \mathbb{R} \to \mathbb{S}^1$ be the function given by $p(x) = e^{2\pi i x}$. One way to visualize p, is to think of $\mathbb{R}$ as being wrapped as a helix inside $\mathbb{R}^3$ and of p as its projection of this helix onto the xy-plane, as shown in Figure 2.1.

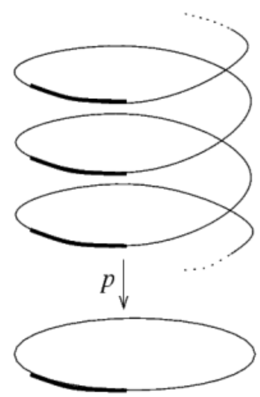


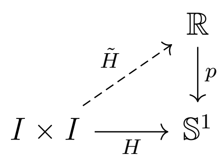
Figure 2.1

For every integer n, define the loop $f_n : I \to \mathbb{S}^1$ given by $x \mapsto e^{2\pi i n x}$, which winds around the circle n times. Note that $[f_n] \cdot [f_m] = [f_{n+m}]$, where the homotopy $f_n \cdot f_m \sim f_{n+m}$ simply reparametrizes the domain (i.e. changes the speed of traversing the loops). We will show that the map $\varphi : \mathbb{Z} \to \pi_1(\mathbb{S}^1, 1)$ given by $n \mapsto [f_n]$ is a group isomorphism. The proof proceeds by studying the lifts with respect to p. For every integer n, define $\tilde{f}_n : I \to \mathbb{R}$, by $\tilde{f}_n(x) = nx$. Clearly, $\tilde{f}_n$ lifts f_n.

We will now show that for any path $u : I \to \mathbb{S}^1$ with $u(0) = 1$, there is a unique path $\tilde{u} : I \to \mathbb{R}$ lifting u such that $\tilde{u}(0) = 0$. Let $S_1 = \mathbb{S}^1 \backslash \{1\}$ and $S_2 = \mathbb{S}^1 \backslash \{-1\}$, and notice that $p^{-1}(S_1)$ consists of a disjoint union of neighborhoods in $\mathbb{R}$ each homeomorphic to S_1, and similarly for S_2. By compactness of I, the open cover given by $u(\{S_1, S_2\})$ contains a finite subcover; so we can choose $N \in \mathbb{N}$ large enough so that for all $1 \leqslant k \leqslant N$ we have $u\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right) \subseteq S_1$ or $u\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right) \subseteq S_2$. Now, we construct $\tilde{u}$ inductively on the intervals $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ as follows. Suppose that $\left[\frac{k-1}{n}, \frac{k}{n}\right] \subseteq S_i$, and let U be the unique neighborhood in $\mathbb{R}$ in $p^{-1}(S_i)$ that contains $\tilde{u}\left(\frac{k-1}{n}\right)$. Then, p maps U homeomorphically onto S_i, and we get the desired lift as

$\tilde{u}\big|_{[\frac{k-1}{n},\frac{k}{n}]} = \big(p\big|_U\big)^{-1} \circ u$. To see that the lift is unique, suppose $\tilde{u}$ and $\tilde{u}'$ are both lifts of u with $\tilde{u}(0) = \tilde{u}'(0) = 0$. Since $p \circ \tilde{u} = p \circ \tilde{u}'$, we conclude that $\tilde{u}(s) - \tilde{u}'(s) \in \mathbb{Z}$ for all $s \in I$. Since $\tilde{u} - \tilde{u}' : I \to \mathbb{Z}$ is a continuous function from a connected space to a discrete space, it must be constant (this is an easy exercise, for a proof see lemma 4.18 of [Cro05]). Hence, $\tilde{u}(s) = \tilde{u}(0) = 0 = \tilde{u}'(0) = \tilde{u}'(s)$ for all $s \in I$.

Define $\psi : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ by $[f] \mapsto \tilde{f}(1)$, the endpoint of the lifted path (which is an integer since $p(\tilde{f}(1)) = f(1) = 1$). We will show that a homotopy of loops H between f and g lifts to a unique homotopy of path $\tilde{H} : I \times I \to \mathbb{R}$ satisfying $\tilde{H}(0,0) = 0$ and $p \circ \tilde{H} = H$. Since $\tilde{H}(1, -) : I \to \mathbb{Z}$ is a continuous map from a connected space to a discrete space, it must be constant; therefore, $\tilde{f}(1) = \tilde{H}(1,0) = \tilde{H}(1,1) = \tilde{g}(1)$, which would prove that ψ is well-defined on homotopy classes of loops.



We will find the unique homotopy lift in a similar fashion to the path lift. Let $H : I \times I \to \mathbb{S}^1$ be a homotopy of loops between f and g. By compactness of $I \times I$, the open cover given by $H^{-1}(\{S_1, S_2\})$ contains a finite subcover; so we can find $N \in \mathbb{N}$ large enough and subdivide $I \times I$ into an $N \times N$ grid of squares of side length $1/N$, each of which is mapped by H into either S_1 or S_2. Again, we proceed to define $\tilde{H}$ inductively on each square, starting with $[0, \frac{1}{N}] \times [0, \frac{1}{N}]$. Suppose it is mapped into S_i. Take V to be the unique component of $p^{-1}(S_i)$ containing $\tilde{H}(0,0) = 0$. We get a unique lift of our first sub-square as $\big(p\big|_V\big)^{-1} \circ H\big|_{[0,\frac{1}{N}]\times[0,\frac{1}{N}]}$. Next, we define $\tilde{H}$ on $[0, \frac{1}{N}] \times [\frac{1}{N}, \frac{2}{N}]$ in a similar fashion using now as initial point $\tilde{H}(0, \frac{1}{N})$. The potential problem of contradicting definitions of $\tilde{H}$ on the shared edge $[0, \frac{1}{N}] \times \frac{1}{N}$ does not occur by the uniqueness of path lifting since both definitions lift the same path and agree (by definition) on $(0, \frac{1}{N})$. Similarly, we continue to lift H on each square using as an initial point a vertex on which $\tilde{H}$ was defined in a previous square. Assuming that we have two lifts $\tilde{H}$ and $\tilde{H}'$ with $\tilde{H}(0,0) = \tilde{H}'(0,0) = 0$, we again conclude that $\tilde{H} - \tilde{H}' : I \times I \to \mathbb{Z}$ is constant since it is a map from a connected space to a discrete space, and the uniqueness follows. Thus, we proved that ψ is well-defined.

Since $\psi(f_n) = \tilde{f}_n(1) = n$, we have $\psi \circ \phi = \mathbb{1}_{\mathbb{Z}}$. It remains to show that ψ is injective, which would show that both ψ and ϕ are bijective. Suppose $\psi([f]) = \psi([g])$ and so $\tilde{f}(1) = \tilde{g}(1)$. Then, $\tilde{g}^- \cdot \tilde{f}$ is a loop in $\mathbb{R}$ based at 0. The homotopy $H : I \times I \to \mathbb{R}$ given by $H(s,t) = (1-t)(\tilde{g}^- \cdot \tilde{f})(s)$ shows that $[\tilde{g}^- \cdot \tilde{f}] = [0]$. By applying the map $p_\# : \pi_1(\mathbb{R}, 0) \to \pi_1(\mathbb{S}^1, 1)$ induced by p to both sides, we get $[g^-] \cdot [f] = [1]$. Therefore, $[g] = [f]$ and the proof is complete. □

Chapter 3

Covering Spaces

The aim of this chapter is to generalize the strategy used in the proof of Theorem 2.6 into a theory for calculating fundamental groups of well-behaved spaces. Besides their central role in homotopy theory, covering spaces play an important role in complex analysis, differentiable manifolds, and more. We shall only introduce the concepts that will be useful for us later on, but the reader may refer to lecture 6 of [FF16], for example, for a more comprehensive introduction to the theory.

Definition 3.1. *A* covering space *(or simply a* covering*) of X is a map $p : \tilde{X} \to X$ satisfying the following: for every $x \in X$ there is an open neighborhood $U \subseteq X$ of x such that $p^{-1}(U)$ is the union of disjoint open sets $\{V_\alpha\}_{\alpha \in A}$, with each V_α homeomorphic to U. The open set U is called* evenly covered*, each V_α is called a sheet above U, and for any point $x \in X$ the set $p^{-1}(x)$ is called the fibers above x.*

Example 3.2. $\mathbb{R}$ is a covering space of $\mathbb{S}^1$ with the map p defined in Theorem 2.6 being the covering map.

Example 3.3. *For every positive integer n,* $\mathbb{S}^1$ is a covering space over itself via the covering map $z \mapsto z^n$. In this case, every evenly covered neighborhood has n sheets above it.

Note that if U is a connected evenly covered neighborhood, the sheets above U are the connected components of $p^{-1}(U)$. The number of sheets above U is the cardinality of $p^{-1}(x)$ where $x \in U$. As x varies over X, this number is constant as long as we are in the same path component, and so it is constant if X is path-connected. In that case, we call it the *degree of the covering.*

3.1 Path and Homotopy Lifting Properties

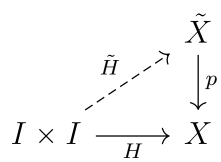
In Theorem 2.6, we proved the existence and uniqueness of path lifts and homotopy lifts for $\mathbb{S}^1$. This section aims to prove these property in a much more general setting. Let us assume throughout the chapter that all spaces are Hausdorff.

Lemma 3.4. (Path Lifting Property). *Let $p : \tilde{X} \to X$ be a covering, $\tilde{x}_0 \in \tilde{X}$, and $u : I \to X$ be a path such that $u(0) = p(\tilde{x}_0) = x_0$. Then, there is a unique lift $\tilde{u} : I \to \tilde{X}$ of u such that $\tilde{u}(0) = \tilde{x}_0$.*

Proof. We have an open cover of $u(I)$ of evenly covered neighborhoods, and so by compactness of I we have a finite open cover of I with each set in this cover mapping inside an evenly covered neighborhood of X. Therefore we can find $N \in \mathbb{N}$ large enough so that there are evenly covered neighborhoods $\{U_k\}_{k=1}^N$ with $u([\frac{k-1}{N}, \frac{k}{N}]) \subseteq U_k$ for $1 \leqslant k \leqslant N$. We construct $\tilde{u}$ inductively as follows. Assume there exists $\tilde{u}_k : [0, \frac{k}{N}] \to \tilde{X}$ with $\tilde{u}_k(0) = \tilde{x}_0$ and $p \circ \tilde{u}_k = u\big|_{[0,\frac{k}{N}]}$. This holds for $k = 0$ by setting $\tilde{u}_k(0) = \tilde{x}_0$. Since U_{k+1} is evenly covered, there is a unique component V of $p^{-1}(U_{k+1})$ containing $\tilde{u}(\frac{k}{N})$. Moreover, p maps V homeomorphically onto U_{k+1}. We define $\tilde{u}_{k+1} : [0, \frac{k+1}{N}] \to \tilde{X}$ by setting $\tilde{u}_{k+1} = \tilde{u}_k$ on $[0, \frac{k}{N}]$ and $\tilde{u}_{k+1} = \left(p\big|_V\right)^{-1} \circ u\big|_{[\frac{k}{N},\frac{k+1}{N}]}$. The required lift is given by $\tilde{u} = \tilde{u}_N$.

To prove uniqueness, suppose towards a contradiction that $\tilde{u}$ and $\tilde{u}'$ are both lifts of u with $\tilde{u}(0) = \tilde{u}'(0) = \tilde{x}_0$. Let $t_0 = \inf\{t \in I \mid \tilde{u}(t) \neq \tilde{u}'(t)\}$. Taking the image under $\tilde{u}$ and under $\tilde{u}'$ of an increasing sequence in I converging to t_0, we see that $\tilde{u}(t_0) = \tilde{u}'(t_0)$ by the uniqueness of the limit in a Hausdorff space. Let U be an evenly covered neighborhood of $u(t_0)$, and let $\tilde{U}$ be the sheet above U containing $\tilde{u}(t_0) = \tilde{u}'(t_0)$. Then, there exists $\varepsilon > 0$ such that $\tilde{u}$ and $\tilde{u}'$ maps $(t_0 - \varepsilon, t_0 + \varepsilon)$ into $\tilde{U}$. Since $p \circ \tilde{u} = p \circ \tilde{u}'$, applying $\left(p\big|_{\tilde{U}}\right)^{-1}$ to both sides, we see that the two lifts agree on an ε-neighborhood of t_0. This contradicts the definition of t_0 as an infimum. □

Theorem 3.5. (Homotopy Lifting Property). *Let $p : \tilde{X} \to X$ be a covering, and $H : I \times I \to X$ be a homotopy of paths. Given a point $\tilde{x}_0 \in \tilde{X}$, there exists a unique homotopy $\tilde{H} : I \times I \to \tilde{X}$ such that $\tilde{H}(0,0) = \tilde{x}_0$ and $p \circ \tilde{H} = H$. Moreover, if $H(s,0)$ and $H(s,1)$ are constant for all $s \in I$ (i.e. all paths share the same endpoints), then $\tilde{H}(s,0)$ and $\tilde{H}(s,1)$ are also constant for all $s \in I$.*



Proof. Take any cover $\mathcal{O}$ of X by evenly covered open sets, and consider the open cover $\{H^{-1}(U) \mid U \in \mathcal{O}\}$ of $I \times I$. It has a finite subcover by compactness of $I \times I$, and so we can find $N \in \mathbb{N}$ large enough so that any square of side length $1/N$ is mapped by H into a neighborhood in $\mathcal{O}$. Define $Q_{i,j} = [\frac{i-1}{N}, \frac{i}{N}] \times [\frac{j-1}{N}, \frac{j}{N}]$ for $1 \leqslant i, j \leqslant N$. We will construct $\tilde{H}$ inductively on each $Q_{i,j}$. For every i, j, take $U_{i,j}$ to be any neighborhood in $\mathcal{O}$ containing $H(Q_{i,j})$. Let V be the unique sheet above $U_{1,1}$ that contains $\tilde{x}_0$, and define $\tilde{H}$ on $Q_{1,1}$ by $(p\big|_{U_{1,1}})^{-1} \circ H\big|_{Q_{1,1}}$. Next, we define $\tilde{H}$ on $Q_{1,2}$ similarly, now using the sheet above $U_{1,2}$ containing $\tilde{H}(0, 1/N)$. By the previous theorem, we don't get a contradicting definition of $\tilde{H}$ on the common edge $[0, \frac{1}{N}] \times \{\frac{1}{N}\}$ since both must be the unique lift of the edge starting at the point $\tilde{H}(0, 1/N)$. We continue in the same fashion, using as initial point any vertex on which $\tilde{H}$ has already been defined, until $\tilde{H}$ is defined on the whole unit square. Uniqueness follows from uniqueness of path lifting since having homotopy lifts that are different at any point implies that a path connecting $(0,0)$ to that point has distinct lifts both starting at $\tilde{x}_0$.

If $p \circ \tilde{H}(s,0) = H(s,0) = x_0$ for all $s \in I$, then the connected set $I \times \{0\}$ is mapped by $\tilde{H}$ into the discrete set $p^{-1}(x_0)$; so $\tilde{H}(s,0) = \tilde{H}(0,0) = \tilde{x}_0$ for all $s \in I$. Similarly, if H is constant on $I \times \{1\}$, then so is $\tilde{H}$. □

As a matter of fact, the homotopy lifting theorem holds in a more general setting than just homotopies of path. More precisely, one can show that unique lifts exist for homotopies of the form $H : Y \times I \to X$ for a sufficiently nice space Y. For our purposes, however, the previous theorem will be sufficient.

3.2 Action of the Fundamental Group on Fibers

The following consequence of the homotopy lifting theorem allows us to identify $\pi_1(\tilde{X}, \tilde{x}_0)$ as a subgroup of $\pi_1(X, x_0)$.

Theorem 3.6. *Given a covering $p : \tilde{X} \to X$, with basepoints $\tilde{x}_0 \in \tilde{X}$ and $x_0 = p(\tilde{x}_0) \in X$, the induced homomorphism $p_\# : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.*

Proof. Let $\tilde{\gamma}$ be a loop that represents an element of the kernel of $p_\#$. So $\gamma = p \circ \tilde{\gamma}$ is homotopic to the constant loop at x_0. By the homotopy lifting theorem, we conclude that $\tilde{\gamma}$ is homotopic to the constant loop at $\tilde{x}_0$ since the latter is a lift of the constant loop at x_0. That is $[\tilde{\gamma}]$ is the identity of $\pi_1(\tilde{X}, \tilde{x}_0)$. □

Definition 3.7. *The* group of a covering $p : \tilde{X} \to X$ *is the subgroup $p_\# \pi_1(\tilde{X}, \tilde{x}_0) \leqslant \pi_1(X, x_0)$, which is isomorphic to the fundamental group of $\tilde{X}$.*

The fundamental group of a space acts naturally on the fibers of its covering. This action provides information about the fundamental group itself. We now turn to define this action and examine its properties.

Theorem 3.8. *Let $p : \tilde{X} \to X$ be a covering, and $x_0 \in X$ be a basepoint. The following description defines a right action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$: given $[\gamma] \in \pi_1(X, x_0)$ and a point $\tilde{x} \in p^{-1}(x_0)$, let $\tilde{x}\,[\gamma] = \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the unique lift of γ starting at $\tilde{x}$.*

Proof. The action is well-defined on homotopy classes of loops by the last sentence in Theorem 3.5. Clearly, the constant loop at $\tilde{x}$ lifts the constant loop at x_0 and so we have $\tilde{x}\,[x_0] = \tilde{x}$. Finally, to show that the action is associative, suppose that γ_1 and γ_2 are loops at x_0 that lift to $\tilde{\gamma}_1$ starting from $\tilde{x}_1 \in p^{-1}(x_0)$ and $\tilde{\gamma}_2$ starting at $\tilde{x}_2 = \tilde{\gamma}_1(1) \in p^{-1}(x_0)$. Note that

$$\tilde{x}_1\left([\gamma_1][\gamma_2]\right) = \tilde{x}_1[\gamma_1 \cdot \gamma_2] = (\widetilde{\gamma_1 \cdot \gamma_2})(1) = \tilde{\gamma}_2(1) = \tilde{x}_2\,[\gamma_2] = (\tilde{x}_1\,[\gamma_1])\,[\gamma_2]$$

□

Theorem 3.9. *Given a covering $p : \tilde{X} \to X$, where $\tilde{X}$ is path-connected, the following properties hold:*

(i) The action defined in Theorem 3.8 is transitive (i.e. for any $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ there exists a loop γ at x_0 such that $\tilde{x}_1[\gamma] = \tilde{x}_2$).

(ii) For any $\tilde{x} \in p^{-1}(x_0)$, the stabilizer of $\tilde{x}$ under the action of $\pi_1(X, x_0)$ is the subgroup $p_\# \pi_1(\tilde{X}, \tilde{x})$ (i.e. $\tilde{x}[\gamma] = \tilde{x}$ if and only if $[\gamma] \in p_\# \pi_1(\tilde{X}, \tilde{x})$).

(iii) The set $\{p_\# \pi_1(\tilde{X}, \tilde{x}) \mid \tilde{x} \in p^{-1}(x_0)\}$ is a conjugacy class of subgroups of $\pi_1(X, x_0)$.

(iv) $|p^{-1}(x_0)| = |\pi_1(X, x_0) : p_\# \pi_1(\tilde{X}, \tilde{x}_0)|$.

Proof. (i) Suppose that $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ and $\tilde{\gamma}$ is a path from $\tilde{x}_1$ to $\tilde{x}_2$. Then, $\gamma = p \circ \tilde{\gamma}$ is a loop at x_0 and so it represents an element in $\pi_1(X, x_0)$. Since $\tilde{\gamma}$ is a lift of γ starting at $\tilde{x}_1$, we see that $\tilde{x}_1[\gamma] = \tilde{x}_2$.

(ii) A loop γ is in the stabilizer of $\tilde{x}$ if and only if its unique lift starting from $\tilde{x}$ ends at $\tilde{x}$ (i.e. it is a loop at $\tilde{x}$). This is indeed the case when $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ and $[\gamma] = p_\#[\tilde{\gamma}]$. Conversely, if $[\gamma] = p_\#[\tilde{\gamma}]$ for some $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$, then a lift of γ starting at $\tilde{x}$ is homotopic to a loop at $\tilde{x}$ by Theorem 3.5.

(iii) is a general fact about transitive group actions. From (i) and (ii), we see that (iv) follows immediately by the Orbit-Stabilizer Theorem. For proofs of these algebraic facts, see chapter 3 of [Rot95] □

The following result illustrates the utility of this theorem in calculating fundamental groups.

Corollary 3.10. *Let $\mathbb{RP}^n$ be the real projective n-space (i.e. the quotient of $\mathbb{S}^n$ by the relation $p \sim -p$, which identifies every two opposite points). Then, for $n > 1$, $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$.*

Proof. The quotient map $p : \mathbb{S}^n \to \mathbb{RP}^n$ is a covering of degree 2. It is not hard to see that $\pi_1(\mathbb{S}^n)$ is trivial for $n > 1$. By part (iv) of Theorem 3.9, we get

$$|p^{-1}(x_0)| = 2 = |\pi_1(\mathbb{RP}^n, x_0) : p_\# \pi_1(\mathbb{S}^n, \tilde{x}_0)|.$$

It follows that $|\pi_1(\mathbb{RP}^n, x_0)| = 2$ □

The fundamental group is different from most of the other homotopy invariants studied in algebraic topology in that it is not always abelian. Define the *wedge sum* $(X, x_0) \vee (Y, y_0)$ of two based spaces X and Y as the quotient space of their disjoint union by the identification of their basepoints $x_0 \sim y_0$. We will show that the fundamental group of $\mathbb{S}^1 \vee \mathbb{S}^1$ is nonabelian.

Corollary 3.11. *Let $i, j : \mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1$ be the two natural embeddings. Let α and β be the images of the generator of $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ under $i_\#$ and $j_\#$. Then $\alpha \cdot \beta \neq \beta \cdot \alpha$.*

Proof. Consider the highlighted lifting of the loop representing $\alpha\beta\alpha^-\beta^-$ in the covering of $\mathbb{S}^1 \vee \mathbb{S}^1$ shown in Figure 3.1. Since this lift is not a loop, it does not represent an element of the group of the covering. Therefore, $\alpha\beta\alpha^-\beta^-$ is not in the stabilizer of $\tilde{x}_0$ by part (ii) of Theorem 3.9. In particular, $\alpha\beta\alpha^-\beta^-$ is a non-trivial element of $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$. □

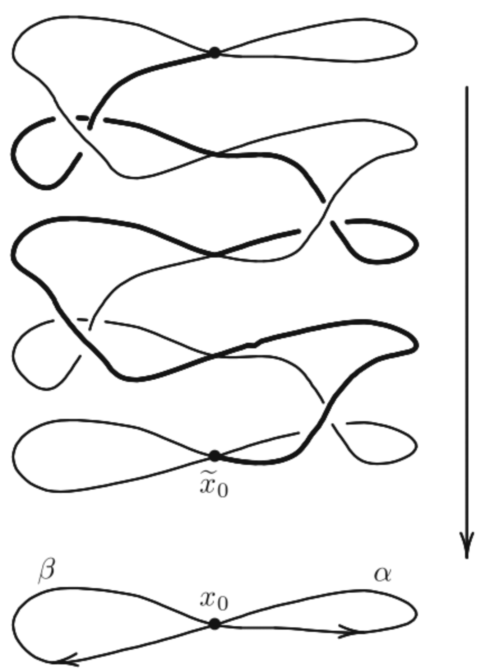


Figure 3.1 a degree 5 covering of $\mathbb{S}^1 \vee \mathbb{S}^1$. Image from [FF16]

3.3 Lifting of General Maps

We have already seen the importance of lifts. It is therefore desirable to understand the necessary and sufficient conditions for the existence of lifts of any continuous function. First we need to define the following property.

Definition 3.12. *A space is* locally path-connected at x *if for every open set* U *containing* x*, there exists a path-connected open set* V *containing* x *with* $V \subseteq U$*. The space is called* locally path-connected *if it is locally path-connected at every point.*

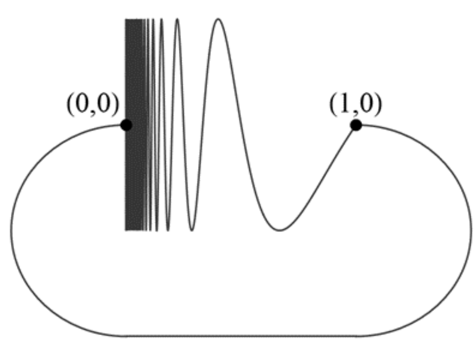
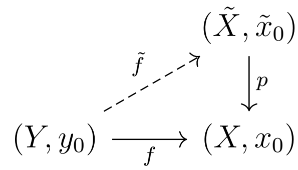


Figure 3.2 Warsaw circle

Any connected, locally path-connected space is also path-connected. Surprisingly, not all path-connected spaces are locally path-connected. One counter example is the Warsaw circle given by closing the curve $y = sin(\pi/x)$ for $0 < x < 1$ (called the topologist's since curve) by an arc from $(0,0)$ to $(1,0)$ (see Figure 3.2). All the spaces we will consider will be path-connected and locally path-connected, so we need not worry too much about these subtleties.

Theorem 3.13. *Let Y be a path-connected, locally path-connected space, $p : \tilde{X} \to X$ a covering of X, and $f : Y \to X$ a continuous function. Given basepoints $\tilde{x}_0 \in \tilde{X}$, $y_0 \in Y$, and $x_0 = f(y_0) = p(\tilde{x}_0) \in X$, a lift $\tilde{f} : Y \to \tilde{X}$ of the map f satisfying $\tilde{f}(y_0) = \tilde{x}_0$ exists if and only if $f_\# \pi_1(Y, y_0) \subseteq p_\# \pi_1(\tilde{X}, \tilde{x}_0)$. In particular, if Y is simply connected, the desired lift always exists. If such a lift exists, it is unique.*



Proof. Assuming that the desired lift exists, since $p \circ \tilde{f} = f$, we have $p_\# \circ \tilde{f}_\# = f_\#$. Therefore,

$$f_\# \pi_1(Y, y_0) = p_\# \left(\tilde{f}_\# \pi_1(Y, y_0) \right) \subseteq p_\# \pi_1(\tilde{X}, \tilde{x}_0).$$

We will now prove the converse. Assume that $f_\# \pi_1(Y, y_0) \subseteq p_\# \pi_1(\tilde{X}, \tilde{x}_0)$. We will define the lift $\tilde{f}$ on an arbitrary point $y \in Y$. Choose any path u from y_0 to y and lift $f \circ u$ to $\widetilde{f \circ u}$ starting at $\tilde{x}_0$. Set $\tilde{f}(y) = \widetilde{f \circ u}(1)$. To show that $\tilde{f}$ is well-defined, consider two paths u_1 and u_2 from y_0 to y. The loop $u_1 \cdot u_2^-$ is mapped by f to a loop γ in representing an element of $f_\# \pi_1(Y, y_0) \subseteq p_\# \pi_1(\tilde{X}, \tilde{x}_0)$. By part (ii) of Theorem 3.9, γ is lifted to a loop at $\tilde{x}_0$ in $\tilde{X}$, which guarantees that $\widetilde{f \circ u_1}$ and $\widetilde{f \circ u_2}$ have the same terminal point.

To see that this lift is unique, assume $\tilde{f}$ and $\tilde{f}'$ both satisfy the requirements of the theorem. Given $y \in Y$, let u be a path from y_0 to y. Since the two path $\tilde{f} \circ u$ and $\tilde{f}' \circ u$ start at $\tilde{x}_0$ and are mapped by p onto $f \circ u$, they must coincide by uniqueness of path lifting. In particular, $\tilde{f}(y) = \tilde{f} \circ u(1) = \tilde{f}' \circ u(1) = \tilde{f}'(y)$.

It remains to show that $\tilde{f}$ is continuous. It is enough to show that for every $y \in Y$ there exists a neighborhood V_y with $\tilde{f}$ being continuous on V_y since then the preimage of an open neighborhood in $\tilde{X}$ would be the union of open sets lying inside the neighborhoods $\{V_y\}_{y \in Y}$ which would be open. Given $y \in Y$, $x = f(y)$, and $\tilde{x} = \tilde{f}(y)$, pick an evenly covered neighborhood U of x and let $\tilde{U}$ be the sheet above it containing $\tilde{x}$. By continuity of f, we can find an open neighborhood $V \subseteq Y$ such that $f(V) \subseteq U$. Since Y is locally path-connected, we may assume that V is path-connected WLOG (otherwise we take V to be a smaller path-connected neighborhood of y). Since $p \circ \tilde{f} = f$, we have $\tilde{f}(V) \subseteq p^{-1}(U)$. It suffices to show that $\tilde{f}(V) \subseteq \tilde{U}$ since we would have $\tilde{f}\big|_V = (p\big|_{\tilde{U}})^{-1} \circ f\big|_V$, which would establish the continuity of $\tilde{f}$. Fix a path u_0 from y_0 to y, and for any $y' \in V$ take a path u from y to y'. Lift $f \circ u_0$ and $f \circ u$ to paths $\widetilde{f \circ u_0}$ starting at $\tilde{x}$ and $\widetilde{f \circ u}$ starting at $\widetilde{f \circ u_0}(1)$, respectively. Since $f \circ u$ lies entirely in U, its lift must lie in $\tilde{U}$, and so

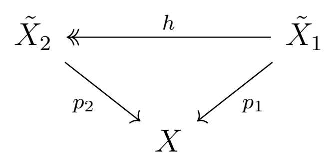
$$f(y') = f \circ \widetilde{(u_0 \cdot u)}(1) = \widetilde{f \circ u}(1) \in \tilde{U}.$$

Finally, if Y is simply connected, then (by definition) $\pi_1(Y, y_0)$ is trivial, and so the condition $f_\# \pi_1(Y, y_0) \subseteq p_\# \pi_1(\tilde{X}, \tilde{x}_0)$ holds. □

3.4 The Fundamental Theorem of Covering Spaces

The following definition is useful for classifying covering spaces.

Definition 3.14. *A* homomorphism of coverings *from $p_1 : \tilde{X}_1 \to X$ to $p_2 : \tilde{X}_2 \to X$ is a continuous surjection h such that $p_2 \circ h = p_1$. If h is a homeomorphism, we call it an* isomorphism of coverings *and we say that the two coverings are equivalent.*



The theory of covering spaces works in its full scope for spaces that satisfy the following property.

Definition 3.15. *A space X is* semi-locally simply connected *if for every point x, there exists a neighborhood U of x such that every loop in U is nullhomotopic in X, i.e. loops in U can be contracted to a point via a homotopy $H : \mathbb{S}^1 \times I \to X$. Equivalently, every point $x \in X$ has a neighborhood U such that the map $\pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion $U \hookrightarrow X$ is trivial.*

The term semi-locally (as opposed to locally) refers to the fact that intermediate loops in the homotopy H may go outside of U. This condition ensures the existence and uniqueness (up to equivalence) of a simply connected covering space (also known as a *universal covering* [1]). For these spaces, the connection between the fundamental group of a space and its coverings is strongest. We summarize the connection, which is reminiscent of Galois theory[2], in the following theorem.

Theorem 3.16. (The Fundamental Theorem of Covering Spaces). *Let (X, x_0) be a path-connected, locally path-connected, and semi-locally simply connected space. There is a bijection between the isomorphism classes of coverings $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$, where $\tilde{X}$ is path-connected on the one hand, and the subgroups of $\pi_1(X, x_0)$ on the other, given by associating to the isomorphism class of p the subgroup $p_\# \pi_1(\tilde{X}, \tilde{x}_0)$. If the basepoints are omitted (i.e. if we think of the same covering with different basepoints as one object), we get a bijection between isomorphism classes of path-connected coverings and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

[1]The term universal refers to the following property: given a universal covering $p : \tilde{X} \to X$ and any other covering $p' : \tilde{X}' \to X$, there exists a (necessarily unique) homomorphism of coverings $\tilde{X} \to \tilde{X}'$. In categorical terms, it is the inital object in the category of coverings of X

[2]This can be formalized as a certain equivalence of categories

Since our focus will be on spaces which are not semi-locally simply connected (also called wild spaces), we will not prove this theorem. The canonical example of a wild space is the *Hawaiian earring*, $\mathbb{H} = \bigcup_{n \geqslant 1} \{(x, y) \in \mathbb{R}^2 \mid (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$, which is not semi-locally simply connected at $(0, 0)$. We will return to the Hawaiian earring and its various realizations in Chapter 6, and it will be the center of our attention for a significant part of this thesis.

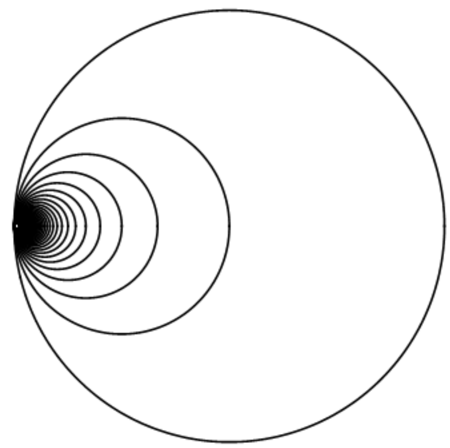


Figure 3.3 the Hawaiian earring

Chapter 4

Seifert–van Kampen Theorem

In this chapter we will present our second important classical tool for calculating the fundamental group of a space, knowing the fundamental group of its constituents. More precisely, the Seifert–van Kampen Theorem (also called van Kampen Theorem) allows us to compute the fundamental group of the union of path-connected subspaces that have a path-connected intersection in terms of the free product of the fundamental groups of these subspaces. We will first briefly define the free product of group and then we will prove van Kampen Theorem and use it to compute the fundamental group of the wedge of circles. For a thorough treatment and more applications of van Kampen Theorem, see [Hat02].

4.1 The Free Product of Groups

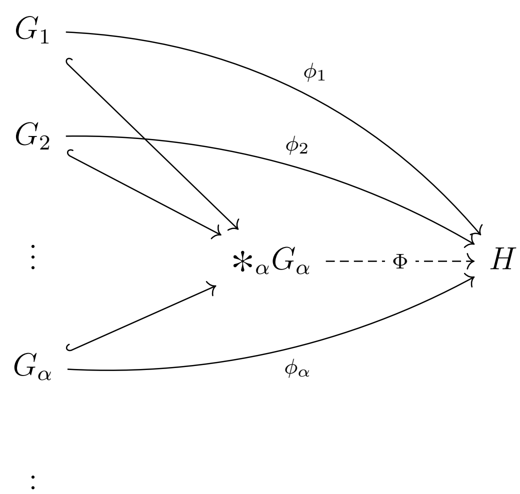
Given groups G and H, a *word* in G and H is a finite sequence of the form $s_1 s_2 \dots s_n$, where each s_i is an element of either G or H. A word may be reduced in one of the following two ways: (1) a consecutive pair of elements $g_i g_{i+1}$ in G (resp. $h_i h_{i+1}$ in H) can be replaced by their product in G (resp. H); and (2) any instance of the identity of either G or H can be removed. By repeated application of these two rules, the reduced words (i.e. those for which neither (1) nor (2) apply) will be of the form of an alternating product $g_1 h_1 g_2 h_2 \dots g_n h_n$ where $g_i \in G$ and $h_i \in G$ for all $1 \leqslant i \leqslant n$ are non trivial elements (with the possible exception of g_1 and h_n).[1] The free product of G and H is defined as the set of reduced words in G and H, with the operation being concatenation followed by reduction, and is denoted by $G * H$. The empty word is assumed to be an element of $G * H$ and serves as its identity. The group axioms are easy to verify. Having defined the free product of two groups, we can easily show that the operation $*$ is associative, and thus one can also define the free product of any finite number of groups by $\ast_{i=1}^{n} G_i = G_1 * G_2 * \cdots * G_n$. More generally, we can analogously define the free product of a collection of groups $\{G_\alpha\}_{\alpha \in A}$ where A is a set of any cardinality. The underlying set of $\ast_\alpha G_\alpha$ is as before the set of (finite) words $g_1 g_2 \dots g_n$ with $g_i \in G_{\alpha_i}$ for some $\alpha_i \in A$, where consecutive letters come from

[1] This is not as obvious as it seems, but we omit the classical proof. For a proof, see chapter 11 of [Rot95].

different groups, i.e. $\alpha_i \neq \alpha_{i+1}$. The group operation is the also the same as in the finite case.

Example 4.1. The free product of n copies of $\mathbb{Z}$. In order not to confuse the elements of the different copies of $\mathbb{Z}$, we can identify the i^{th} copy of $\mathbb{Z}$ with the set $\{\dots, x_i^{-2}, x_i^{-1}, 1, x_i, x_i^2, \dots\}$ in the obvious way. After this identification, we see that an element of the group $*_{i=1}^n \mathbb{Z}$ is a word in the alphabet $\{x_1, x_2, \dots, x_n\}$ and their inverses and powers, where reducing a word looks like regular multiplication of algebraic variables. This group belongs to an important family called the free groups on $|A|$ generators, which we denote by $F_{|A|} = *_{\alpha \in A}\mathbb{Z}$, where the indexing set A is of any cardinality.

The following *universal property* holds of the free product. A collection of homomorphisms $\{\phi_\alpha : G_\alpha \to H\}_\alpha$ extends uniquely to a homomorphism $\Phi : *_\alpha G_\alpha \to H$. Namely, a word $g_1 g_2 \dots g_m$ where $g_i \in G_{\alpha_i}$ is mapped by Φ to the product $\phi_{\alpha_1}(g_1)\phi_{\alpha_2}(g_2) \dots \phi_{\alpha_m}(g_m)$ in H. One can verify that this gives a well-defined homomorphism of groups, and that this property characterizes a unique (up to isomorphism) group isomorphic to the construction presented above. We can restate this property in the commutativity of the following diagram.



4.2 The Theorem

Theorem 4.2. (van Kampen). *Let X be the union of path-connected open sets A_α each containing the basepoint x_0, where each pairwise intersection $A_\alpha \cap A_\beta$ is path connected. Let $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ be the unique homomorphism extending the maps $\pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ induced by the inclusions $A_\alpha \hookrightarrow X$. Then, Φ is surjective.*
Let $f_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta, x_0) \to \pi_1(A_\alpha, x_0)$ be the map induced by inclusion. If in addition every intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the smallest normal subgroup N containing elements of the form $f_{\alpha\beta}(\omega) f_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$. Therefore, $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$. [2]

[2] This construction is known as the amalgamated free product of $\pi_1(A_\alpha)$ with respect to the maps

Proof. First, we will show that Φ is surjective. Let $f : I \to X$ be a loop based at x_0. Then, the preimages under f of the open sets A_i form an open cover of the interval. So by compactness of, there is a subdivision $0 = s_1 < s_2 < \cdots < s_k = 1$ of I such that, for each j, $f([s_{j-1}, s_j]) \subseteq A_{\alpha_j}$ for some α_j. Let f_j be the restriction of f to $[s_{j-1}, s_j]$. For each j, fix a path g_j in $A_{\alpha_j} \cap A_{\alpha_{j+1}}$ from x_0 to $f(s_j)$. Notice that f is homotopic to the loop $(f_1 \cdot g_1^-) \cdot (g_1 \cdot f_2 \cdot g_2^-) \cdot \cdots \cdot (g_{k-2} \cdot f_{k-1} \cdot g_{k-1}^-) \cdot (g_{k-1} \cdot f_k)$, and that each factor $(g_{j-1} \cdot f_j \cdot g_j^-)$ is a loop representing an element of $\pi_1(A_{\alpha_j}, x_0)$. Therefore, this representation of f gives a (possibly unreduced) word of $*_\alpha \pi_1(A_\alpha, x_0)$, and so $[f] \in \operatorname{Im} \Phi$.

By a *factorization* of $[f]$ we mean a representation of $[f] \in \pi_1(X, x_0)$ as a formal product of homotopy classes of loops $[f_1] \ldots [f_k]$, where each $[f_j]$ is an element of $\pi(A_{\alpha_j}, x_0)$ for some α_j, and where $f \sim f_1 \cdot \cdots \cdot f_k$. We say that two factorizations are equivalent if it is possible to get from one factorization to the other by applying a finite sequence of the following two moves and their inverses:

- combine two consecutive classes $[f_i][f_{i+1}]$ into a single class $[f_j \cdot f_{j+1}]$ if both classes are in the same $\pi_1(A_\alpha)$;
- view a class $[f_j]$ as an element of $\pi_1(A_\alpha)$ instead of $\pi_1(A_\beta)$ if f_j is a loop in $A_\alpha \cap A_\beta$.

Notice that applying the first move does not change the reduced word in $*_\alpha \pi_1(A_\alpha)$ represented by the factorization, and applying the second move does not change its coset in $*_\alpha \pi_1(A_\alpha)/N$. Therefore, to prove the second claim, it suffices to show that different factorizations of $[f]$ are equivalent.

Let $[f_1][f_2] \ldots [f_k]$ and $[f_1'][f_2'] \ldots [f_k']$ be two factorizations of $[f]$ (we can assume WLOG that they both contain the same number of factors since we are free to add some trivial classes to make them equal). Since $f_1 \cdot \cdots \cdot f_k \sim f \sim f_1' \cdot \cdots \cdot f_k'$, there is a homotopy $H : I \times I \to X$ of loops in X from $f_1 \cdot \cdots \cdot f_k$ to $f_1' \cdot \cdots \cdot f_k'$. By compactness of $I \times I$, we can find a subdivision $0 = s_1 < s_2 < \cdots < s_p = 1$ and a subdivision $0 = t_1 < t_2 < \cdots < t_q = 1$ so that $H([s_{i-1}, s_i] \times [t_{j-1}, t_j])$ lies inside a single neighborhood A_α. Index the rectangles $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ as $R_1, R_2, \ldots, R_{pq}$ from left to right, then from bottom to top. Notice that each homotopy class in the first factorization of $[f]$ is represented by the restriction of H to some consecutive edges on the bottom side of $I \times I$, and similarly for the second factorization with the top side.

For a vertex (s_i, t_j) in our grid of $I \times I$, we want to take a path from x_0 to the image under H of that vertex. The problem is that $H(s_i, t_j)$ might be contained in the intersection of four different neighborhoods – each of which containing the image of one of the rectangles that share this vertex – which might not be path-connected. This is not a problem, however, since slightly perturbing the edges in the grid can make each vertex the meeting point of at most three rectangles, as shown in Figure 4.1 (we can ensure that our subdivisions were fine enough so that each perturbed rectangle still map into a single neighborhood). Thus, we use our hypothesis that

f_{ij} and f_{ji}, and it is their pushout in the category of groups. Thus, van Kampen Theorem provides sufficient conditions for the preservation of pushouts under the functor π_1

triple intersections are path-connected to get a path u_v from x_0 to $H(v)$ for each vertex v.

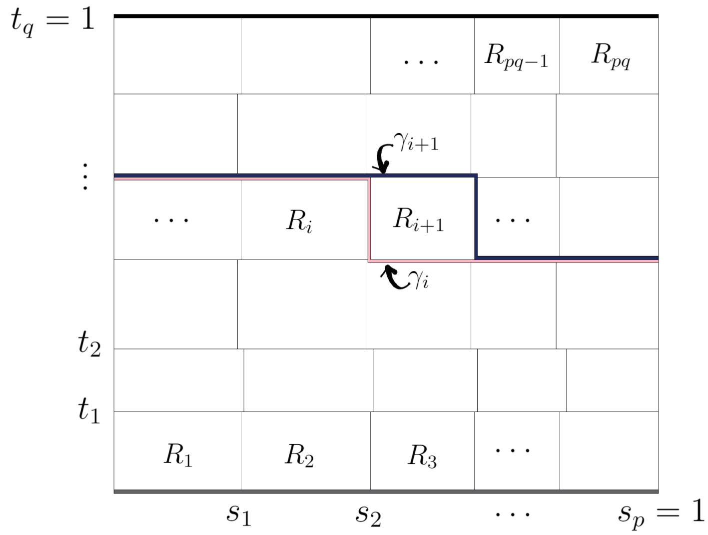


Figure 4.1 the domain of H

Let γ_i be the loop defined by the restriction of H to the path in our grid of $I \times I$ that connects the left side to the right side and separates $R_1, \dots, R_i$ from the other rectangles of the grid, as in Figure 4.1. We can factorize $[\gamma_i]$ as follows. First, notice that we can express γ_i as a concatenation of paths given by its evaluation on the edges of the grid on which it is defined, and each such path stays inside a single neighborhood A_α. Given an edge e of the grid that starts at the vertex v and ends at v', let $\gamma_i'\big|_e = u_v \cdot \gamma_i\big|_e \cdot u_{v'}^-$, which gives a factorization of $[\gamma_i'] = [\gamma_i]$ since the image of any edge under γ_i' stays inside a single A_α.

Notice that the factorization of $[\gamma_0]$ (resp. $[\gamma_{pq}]$) is clearly equivalent to the first (resp. second) factorization of $[f]$. Moreover, the intermediate loops γ_i for $0 < i < pq$ are all homotopic to f in X and define intermediate factorizations of it. Therefore, it is enough to see that for every i the factorizations we get for $[\gamma_i]$ and $[\gamma_{i+1}]$ are equivalent. Observe that γ_i is homotopic to γ_{i+1} via a homotopy inside a single neighborhood A_{α_i} for some α_i, which is given by $H\big|_{R_{i+1}}$. We apply the second move to view the factors of $[\gamma_i]$ and $[\gamma_{i+1}]$ corresponding to the edges of R_{i+1} as elements of $\pi_1(A_{\alpha_i}, x_0)$; then, we apply the first move to combine all the factors corresponding to edges in R_{i+1} into a single factor; and finally we apply the homotopy $H\big|_{R_{i+1}}$ to see that the factors we get from both factorizations is indeed the same element of $\pi_1(A_{\alpha_i}, x_0)$. This shows that the factorizations of $[\gamma_i]$ and of $[\gamma_{i+1}]$ are equivalent and the proof is complete. □

Corollary 4.3. *If in addition to the hypotheses of Theorem 4.2 every pairwise intersection $A_\alpha \cap A_\beta$ is simply connected, then $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)$.*

4.3 Applications

Example 4.4. We can use van Kampen theorem to compute the fundamental group of the wedge of circles $\bigvee_{\alpha \in J} \mathbb{S}^1_\alpha$, where every $\mathbb{S}^1_\alpha$ is an identical copy of the circle with a marked basepoint x_α. Let x_0 be the common basepoint after identifying all the points x_α in the wedge. For every $\alpha \in J$, let U_α be a contractible neighborhood of x_α in $\mathbb{S}^1_\alpha$, and let $A_\alpha = \mathbb{S}^1_\alpha \cup \left(\bigcup_{\beta \in J \setminus \{\alpha\}} U_\beta \right)$. Each A_α is path-connected and open in the wedge of circles, and their union covers the wedge; additionally, the intersection of any two or more distinct sets of $\{A_\alpha\}_{\alpha \in J}$ is $\bigcup_{\alpha \in J} U_\alpha$ which is also path-connected. Therefore, the hypotheses of van Kampen theorem are satisfied. Notice that $\pi_1 \left(\bigcup_{\alpha \in J} U_\alpha, x_0 \right) \cong 0$, and $\pi_1(A_\alpha, x_0) \cong \mathbb{Z}$. It follows that $\pi_1((\bigvee_{\alpha \in J} \mathbb{S}^1) \cong *_{\alpha \in J} \mathbb{Z} \cong F_{|J|}$, the free group on $|J|$ generators.

A graph can be realized as a topological space as follows. A *topological graph* is a space the consists of a set of distinct points (vertices) and a set of homeomorphic images of the unit interval (edges), with each edge glued to two vertices from its endpoints. A tree is a connected graph that contains no closed loops. A *maximal tree* (also called a *spanning tree*) in a graph is a tree that is not properly contained in any other tree in the graph. It is a standard result in graph theory that any connected graph has a maximal tree, and that such a tree contains every vertex of the graph (for a proof and an algorithm to construct maximal trees, see section 1.3.3 of [HHM08]. Van Kampen theorem gives us the following characterization of the fundamental groups of graphs.

Theorem 4.5. *Let X be a connected graph and $T \subseteq X$ a maximal tree. Then, $\pi_1(X) \cong F_n$, where n is the number of edges in $X \setminus T$.*

Proof. Since T contains every vertex, the quotient X/T collapses T into a point and each edge in $X \setminus T$ into a cycle. Thus, X/T is homeomorphic to $\bigvee_{i=1}^n \mathbb{S}^1$. By proving that the quotient map $q : X \to X/T$ is a homotopy equivalence, the theorem will follow from example 4.4 and the fact that the fundamental group is a homotopy invariant. Fix a vertex v_0 in X, and for each other vertex v_i, let u_i be a path from v_0 to v_i in T. Define a map $s : X/T \cong \bigvee \mathbb{S}^1 \to X$ by sending a circle corresponding to the edge e from a vertex v_1 to v_2 to the concatenation $u_1 \cdot e \cdot u_2^-$. The composition $q \circ s$ on any cycle in the wedge looks like $v_0 \cdot \mathbb{1}_{\mathbb{S}^1} \cdot v_0$, which is homotopic to the identity. We can find a deformation retraction of T contracting it to v_0 and thus the two maps $s \circ q\big|_T$ and $\mathbb{1}_T$ are homotopic. We can then extend such a homotopy on all of X by specifying a homotopy from each edge e in $X \setminus T$ with endpoints u_i and u_j to $u_i \cdot e \cdot u_j^-$, which is easily done. ∎

Chapter 5

Interlude on Inverse and Direct Limits

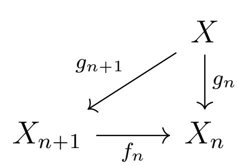
In this chapter, we will describe two important constructions: the inverse limit (or projective limit) and the direct limit (or injective limit).[1] These constructions are characterized by the same *universal properties* for sets, groups, and topological spaces. To avoid repetition, we will sometimes refer sets, groups, and topological spaces by the generic word “object”, where the maps between the objects depend on its type: functions for sets, homomorphisms for groups, and continuous functions for topological spaces.

5.1 Universal Properties

An *inverse system* is a sequence of objects $\{X_n\}_{n\in\mathbb{N}}$ and maps $f_n : X_{n+1} \to X_n$. [2]

$$\ldots \xrightarrow{f_{n+1}} X_{n+1} \xrightarrow{f_n} X_n \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1$$

The *inverse limit* of this system, denoted as $\varprojlim_n X_n$, is an object X together with *projection maps* $g_n : X \to X_n$ such that for every $n \in \mathbb{N}$, the diagram

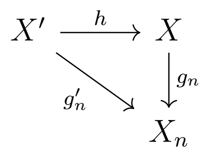


commutes, and it is *universal* in the following sense: for any other object X' with maps $g'_n : X' \to X_n$ satisfying $f_n \circ g'_{n+1} = g'_n$ for all n, there exists a unique map

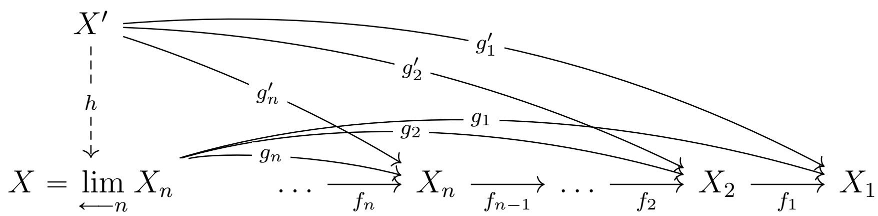
[1] Confusingly enough, in the general setting of category theory an inverse limit is an example of a limit, whereas a direct limit is an example of a colimit.

[2] What we describe here is in fact a special case of a more general construction of inverse limits in which the indexing set of the system need not be $\mathbb{N}$, but can any partially ordered set.

$h : X' \to X$ making the the diagram



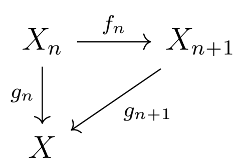
commute for every n. In other words, the pair $(X, \{g_n\}_{n\in\mathbb{N}})$ is the *terminal* one among all such objects with compatible maps. We can summarize the definition in the following commutative diagram.



The notions of a direct system and a direct limit are defined similarly but with all arrows reversed (we call these constructions *duals* for this reason). A *direct system* is a sequence of objects $\{X_n\}_{n\in\mathbb{N}}$ and maps $f_n : X_n \to X_{n+1}$.

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \dots$$

The *direct limit* of this system, denoted as $\varinjlim_n X_n$, is an object X together with maps $g_n : X_n \to X$ such that for every $n \in \mathbb{N}$, the diagram



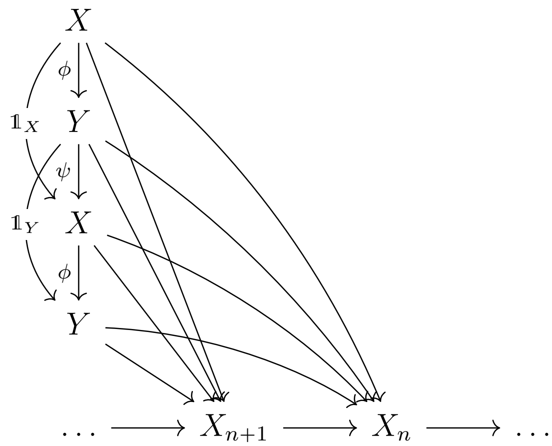
commutes for all n, and it is universal among such objects; i.e. given another object X' with maps $g'_n : X_n \to X'$ satisfying $g'_{n+1} \circ f_n = g'_n$ for all n, there exists a unique map $h : X \to X'$ such that $h \circ g_n = g'_n$.

The direct and inverse limits, if they exist, are unique up to a unique isomorphism making the obvious diagram commute. By an isomorphism we mean a map $\phi : X \to Y$ such that there exists a map $\psi : Y \to X$ with $\phi \circ \psi = \mathbb{1}_Y$ and $\psi \circ \phi = \mathbb{1}_X$, which generalizes a bijection of sets, an isomorphism of groups, and a homeomorphism of topological spaces. We will show the uniqueness for the inverse limit and the proof for the direct limit is analogous.

Proposition 5.1. *Let X and Y, with projection maps $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ respectively, be inverse limits of the inverse system $\dots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1$. Then, there is a unique isomorphism $\phi : X \to Y$ such that $h_n \circ \phi = g_n$ for all $n \in \mathbb{N}$.*

Proof. By the universal property of Y, we have a unique map $\phi : X \to Y$ satisfying the required commutativity condition. Similarly, by the universal property of X, we have a unique map $\psi : Y \to X$ such that $g_n \circ \psi = h_n$ for each n. We will show that these are mutual inverses.

Notice that both of the maps $\psi \circ \phi$ and $\mathbb{1}_X$ are solutions for s in $g_n \circ s = g_n$. Considering the uniqueness of the map in the universal property when taking both X and X' to be X with projection maps g_n, we conclude that $\psi \circ \phi = \mathbb{1}_X$. Similarly, $\phi \circ \psi$ and $\mathbb{1}_Y$ are both solutions for t in $h_n \circ t = h_n$, and so by uniqueness again we get $\phi \circ \psi = \mathbb{1}_Y$.



□

5.2 Inverse and Direct Limits of Sets, Groups, and Topological Spaces

Although the universal properties uniquely characterize the inverse and direct limits (when they exist) in a very general setting, it is useful to understand the "microscopic" structure of these objects in the specific context of sets, groups and topological spaces. By giving their explicit construction, we will have also proved that when our objects are sets, groups, or topological spaces, the inverse and direct limits always exist.

Given an inverse system of sets $\ldots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1$, the inverse limit of this system is given by

$$\varprojlim_n X_n = X := \left\{ (x_1, x_2, \ldots) \in \prod_{n \in \mathbb{N}} X_n \,\middle|\, x_i = f_i(x_{i-1}) \ \forall i \in \mathbb{N} \right\},$$

with projections given by the restrictions of the natural projections $p_i : \prod_{n \in \mathbb{N}} X_n \to X_i$ to X. By uniqueness of the inverse limit, it is enough to verify that X satisfies the universal property. Assume there is a set Y and projections $\{g_i : Y \to X_i\}_{i \in \mathbb{N}}$. Then, the mapping $y \mapsto (g_1(y), g_2(y), \ldots)$ is the unique function $h : Y \to X$ satisfying $p_i\big|_X \circ h = g_i$ for every $i \in \mathbb{N}$.

The same construction gives the underlying sets of the inverse limit of groups and of topological spaces. In the case of groups, X is viewed as a subgroup of the infinite direct product $\prod_{n\in\mathbb{N}} X_n$ (whose underlying set is the infinite cartesian product and in which the operation is component-wise multiplication). It can be easily verified that the functions described for sets are homomorphisms. In the case of topological spaces, X is viewed as a subspace of the product $\prod_{n\in\mathbb{N}} X_n$ equipped with the product topology and is topologized as such. Again, the maps defined for sets are continuous in this case, and so the universal property holds. Equivalently, one can equip X with the coarsest topology making the projection maps continuous (called the *initial topology*).

Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots$ be a direct system of sets, and for every $i < j$ define $f_{i,j} : X_i \to X_j$ by $f_{i,j} = f_j \circ f_{j-1} \circ \cdots \circ f_i$. The direct limit of this system is

$$\varinjlim_n X_n = X := \coprod_{n\in\mathbb{N}} X_n \Big/ \sim,$$

the quotient of the disjoint union of the sets X_n by the equivalence relation given by $x_i \sim x_j$ for $x_i \in X_i$ and $x_j \in X_j$ if and only if there is some $k \in \mathbb{N}$ such that $f_{i,k}(x_i) = f_{j,k}(x_j)$. In other words, elements that eventually map to the same image are identified. The maps $X_i \to X$ are the maps that send an element to its equivalence class, which clearly satisfy the commutativity requirement. Moreover, if a set Y is given along with maps $\{g_i : X_i \to Y\}_{i\in\mathbb{N}}$ satisfying the commutativity condition, then we can define a map $h : X \to Y$ on each equivalence class by selecting any representative from that class, say $x_i \in X_i$ and setting $h([x_i]) = g_i(x_i)$. This is well-defined by the compatibility of the maps g_i with the direct system. Thus, the universal property holds for X.

Again, the same construction works as the underlying sets of the direct limit of topological spaces. X is given the quotient topology on the disjoint union of the spaces X_n with respect to the equivalence relation $\sim$. Equivalently, the topology on X is the finest topology making the maps $X_i \to X$ continuous for all $i \in \mathbb{N}$ (called the *final topology*). For groups, it is almost the same construction except that now, instead of the disjoint union, we have the free product; [3] i.e. $\varprojlim_n X_n = *_{n\in\mathbb{N}} X_n \Big/ \sim$, with the same equivalence relation $\sim$. If $x_i \in X_i$ and $x_j \in X_j$, multiplication is defined by $[x_i][x_j] = [f_{i,k}(x_i) f_{j,k}(x_j)]$ for any $k \geqslant i, j$. Well-definedness can be easily checked, and the group axioms are inherited from those of the groups X_n. Under this group structure, verifying that the maps described for sets are homomorphisms is also straightforward.

[3] Notice that the universal property for the free product described earlier precisely characterizes disjoint unions when groups and homomorphisms are replaced by sets and functions. Both concepts are manifestations of the same categorical construction called *coproduct*. Coproducts are the duals to products, and quotient objects are duals to sub-objects, which clarifies the connection between the seemingly unrelated constructions for the dual notions of direct and inverse limits.

Chapter 6

Non-Semilocally Simply Connected Spaces

Recall that a space is called wild if it is not semi-locally simply connected at one or more of its points (see definition 3.15), i.e. if there is a point in the space from which it is possible to construct arbitrarily small loops that are not nullhomotopic. In this section, we will study some examples of wild spaces and their degree of wildness.

6.1 The Hawaiian Earring

Our first example of a wild space is the Hawaiian earring. Define the circle $C_n = \{(x, y) \in \mathbb{R}^2 \mid (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$ centered at $(1/n, 0)$ with radius $1/n$. The Hawaiian earring is then defined as the union $\mathbb{H} = \bigcup_{n \geqslant 1} C_n$ of all these circles, which are mutually tangent at $x_0 = (0, 0)$ (see Figure 3.3). We give $\mathbb{H}$ the subspace topology from $\mathbb{R}^2$. At first glance, one might think that this space is homeomorphic to a countable wedge of circles. One way to see that this is not true is to note that $\mathbb{H}$ is a closed and bounded subset of $\mathbb{R}^2$, and is therefore compact by Heine-Borel Theorem; whereas the countably infinite wedge of circle is not compact since, for example, taking an open cover in which each open set covers one full circle and a small neighborhood of the basepoint of every other circle, we easily see that it has no finite subcover. Indeed, we know from example 4.4 that the fundamental group of the countable wedge is $\ast_{n \in \mathbb{N}} \mathbb{Z}$, while the fundamental group of the earring will turn out to be a much more intricate object.

Another way to define the Hawaiian earring is via inverse limits. Let $X_n = \bigcup_{k=1}^{n} C_k$. For each $n \in \mathbb{N}$, there is a retraction $r_n : X_{n+1} \to X_n$ that collapses the circle C_{n+1} to x_0 and acts as the identity on the circles C_k for $k \leqslant n$. This gives the following inverse system.

$$\dots \xrightarrow{r_{n+1}} X_{n+1} \xrightarrow{r_n} X_n \xrightarrow{r_{n-1}} \dots \xrightarrow{r_2} X_2 \xrightarrow{r_1} X_1$$

Proposition 6.1. $\mathbb{H} \cong \varprojlim_n X_n$.

Proof. Let $f_k : \mathbb{H} \to X_k$ be the map obtained by collapsing all circles in $\mathbb{H} \backslash (C_1 \cup$

$C_2 \cup \cdots \cup C_k$) to x_0, and maps each C_i for $1 \leqslant i \leqslant k$ homeomorphically via a map h_i into the corresponding circle. These maps are compatible with the inverse system, i.e. $r_{k-1} \circ f_k = f_{k-1}$ for every k. Therefore, by the universal property, we have a unique map $f : \mathbb{H} \to \varprojlim_n X_n$ making all the obvious triangles commute. Notice that if $x \in \mathbb{H}$ is a point on the n^{th} circle, we have

$$f(x) = (x_0, x_0, ..., x_0, h_n(x), h_n(x), ...) \in \varprojlim_n X_n \subset \prod_{n \geqslant 1} X_n,$$

where the first occurrence of $h_n(x)$ is at the n^{th} position. Assume that $f(x) = f(y)$ for some $x, y \in \mathbb{H}$ By comparing the tuples we conclude that x and y lie in the same circle, say the n^{th} circle, and that $h_n(x) = h_n(y)$. Thus, $x = y$ since h_n is a homeomorphism. Moreover, every element of $\varprojlim_n X_n$ is a similar tuple and so it is in the image of f. Therefore, f is a continuous bijection. Since the inverse limit is a subspace of the product, and since the product of Hausdorff spaces is Hausdorff and a subspace of a Hausdorff space is Hausdorff, $\varprojlim_n X_n$ must be Hausdorff as an inverse limit of Hausdorff spaces. Since f is a continuous bijection from a compact space to a Hausdorff space, it must be a homeomorphism (for a proof, see Theorem 5.24 of [Cro05]). □

The topology on $\mathbb{H}$ is coarser than that of $\bigvee_{n \geqslant 1} \mathbb{S}^1$. Indeed, there are natural maps from the wedge to each of the spaces X_n that are compatible with the inverse system, and so we get a continuous map $\bigvee_{n \geqslant 1} \mathbb{S}^1 \to \mathbb{H}$ by the universal property of the inverse limit.

The Hawaiian earring is one of the simplest wild spaces in a sense: it only fails to be semi-locally simply connected at a single point. The following definition due to Brazas is useful for studying the degree of wildness of a space.

Definition 6.2. [Bra20] *Given a space X, the* topological 1-wild set *of X is defined as* $\mathbf{w}(X) = \{x \in X \mid X \text{ is not semi-locally simply connected at } x\}$ *equipped with the subspace topology.*

In this notation, we write $\mathbf{w}(\mathbb{H}) = \{x_0\}$.

6.2 Related Spaces

Once we have the Hawaiian earring, we can define other wild spaces in various ways. The *double Hawaiian earring*, for instance, is a space constructed as follows. Let C_n and D_n be the circles of radius $1/n$ centered at $(\frac{1}{n}, 0)$ and $(-1 - \frac{1}{n}, 0)$, respectively. Then the double earring is defined as $\mathbb{DE} = \bigcup_{n \geqslant 1} C_n \cup ([-1, 0] \times \{0\}) \cup \bigcup_{n \geqslant 1} D_n$. Its topological 1-wild consists of the two points $(-1, 0)$ and $(0, 0)$.

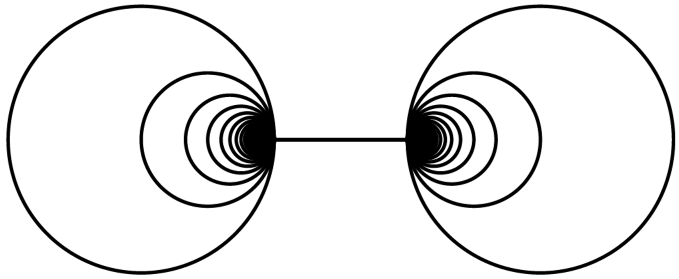


Figure 6.1 the double Hawaiian earring

Consider the open path-connected sets $U = \bigcup_{n\geqslant 1} C_n \cup ((-1, 0] \times \{0\})$ and $V = \bigcup_{n\geqslant 1} D_n \cup ([-1, 0) \times \{0\})$ with the basepoint $(-\frac{1}{2}, 0)$. By applying van Kampen theorem, we see that $\pi_1(\mathbb{DE}) = \pi_1(\mathbb{H}) * \pi_1(\mathbb{H})$.

One can construct much more complicated spaces using the earring. The *wild circle*, $\mathbb{W}$, is defined by attaching a copy of $\mathbb{H}$ at each point of $\mathbb{S}^1$ as shown in Figure 6.2. Note that $\mathbf{w}(\mathbb{W}) = \mathbb{S}^1$ with its usual topology. Given compact, connected, locally connected metric spaces X and Y, it was shown by Eda that if $\mathbf{w}(X)$ and $\mathbf{w}(Y)$ are not homotopically equivalent, then X and Y themselves are not homotopy equivalent [Eda10]. So each of the three spaces we described so far are of different homotopy types. The proof of this theorem, however, is beyond the scope of this thesis.

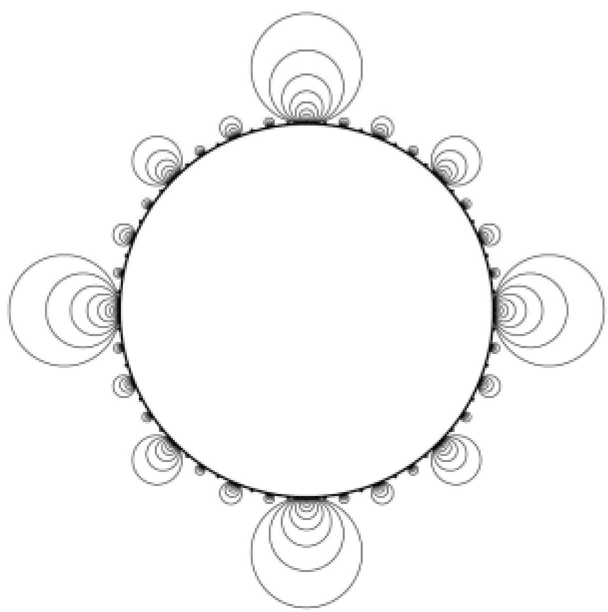


Figure 6.2 the wild circle

There are wild spaces in higher than one dimension as well. One such space is the *harmonic archipelago*, $\mathbb{HA}$. Consider the disc $D \subset \mathbb{R}^3$ of radius 1/2 and centered at $(1/2, 0, 0)$. Draw a copy of the Hawaiian earring on D, and let C_n denote the n^{th} circle of the earring, as before. For each $n \in \mathbb{N}$ take a small disc E_n between C_n and C_{n+1} and push it upwards to make a "hill" of height 1. More formally, if K_n is a (smooth) cone of height 1 above E_n, then $\mathbb{HA} = D \setminus \left(\bigcup_{n\geqslant 1} \mathrm{Int}(E_n)\right) \cup \left(\bigcup_{n\geqslant 1} K_n\right)$. This space is different from all the previous spaces in that it is not compact; indeed, if the top point of the n^{th} hill is denoted by a_n, then $\lim_{n\to\infty} a_n = (0,0,1) \notin \mathbb{HA}$. Although it may seem at first glance that $\mathbb{HA}$ is contractible, the compactness of $I \times I$ will necessitate that no homotopy deforms a loop over infinitely many hills, as we will show in Chapter 8. The harmonic archipelago is thus not semi-locally simply connected at $(0,0,0)$, and $\mathbf{w}(\mathbb{HA}) = \{(0,0,0)\}$.

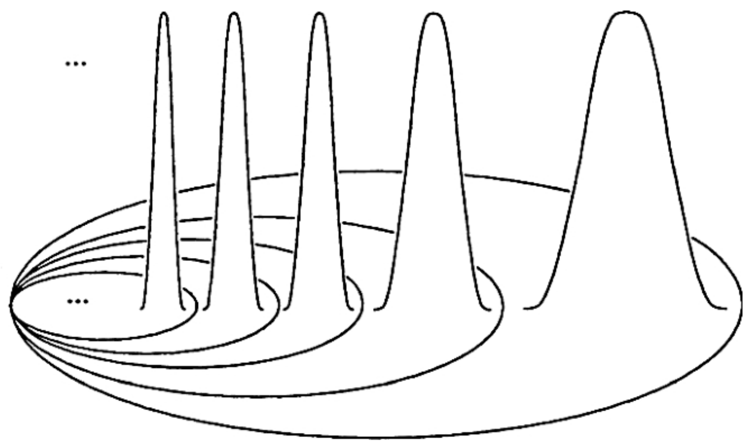


Figure 6.3 the harmonic archipelago

Chapter 7

The Fundamental Group of the Hawaiian Earring

In chapters 3 we saw the importance of universal coverings in calculating fundamental groups. The Hawaiian earring, however, possesses no universal covering. Last chapter, we expressed the earring as an inverse limit of an approximating sequence of semi-locally simply connected spaces, each of which has a universal covering. In this chapter, we will be able to use these coverings to construct a space that "almost covers" $\mathbb{H}$, in the sense that it would enjoy many of the lifting properties that universal covers have. This will allow us to realize the fundamental group of the Hawaiian earring (henceforth referred to as the Hawaiian group) as a subgroup of the inverse limit of fundamental groups of the approximating spaces. As a result, we will be able to give a combinatorial description of the $\pi_1(\mathbb{H})$ in terms of "transfinite words". As we will see, this approach easily generalizes to many other 1-dimensional spaces. The approach we take is due to Eda and Kawamura [EK98] building on the ideas of Curtis and Fort [CF59], and an exposition of it can be found [Bra13, Bra19a, Bra19b].

7.1 Wild Loops

For every $n \geqslant 1$, denote by $\ell_n : I \to \mathbb{H}$ the loop that traverses the n^{th} circle of the earring. Intuitively, it seems that the homotopy classes of the loops ℓ_n should generate the Hawaiian group. But these are not generators in the standard sense due to the fact that we can have infinite words in the letters $[\ell_n]$. For example, the map $\gamma : I \to \mathbb{H}$ defined on the interval $[\frac{n-1}{n}, \frac{n}{n+1}]$ for each $n \geqslant 1$ to be the loop traversing the n^{th} circle of $\mathbb{H}$ and $\gamma(1) = x_0 = (0,0)$ gives us a loop in $\mathbb{H}$. It seems that the homotopy class $[\gamma] = [\ell_1 \cdot \ell_2 \cdot \ell_3 \cdot \ldots]$ does not belong to the free subgroup in $\pi_1(\mathbb{H})$ generated by $\{[\ell_1], [\ell_2], \ldots\}$. Indeed, the following proposition, along with the fact that the free group on countable generators is countable, shows that $\pi_1(\mathbb{H})$ is vastly more complicated.

Proposition 7.1. $\pi_1(\mathbb{H})$ *is uncountably generated.*

Proof. It suffices to show that $\pi_1(\mathbb{H})$ is uncountable. Define the direct product $\prod_\alpha G_\alpha$

of groups G_α to be their cartesian product with the operation of component-wise multiplication. We will prove the theorem by finding an injection from the uncountable group $\prod_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ into $\pi_1(\mathbb{H})$. Given a sequence $s = (a_n) \in \mathbb{Z}/2\mathbb{Z}$, define a loop $\gamma_s : I \to \mathbb{H}$ by setting $\gamma_s(1) = x_0$ and letting the image of $[\frac{n-1}{n}, \frac{n}{n+1}]$ be constant at x_0 if $a_n = 0$, and be the loop traversing the n^{th} circle of the earring otherwise. Given $s = (a_n) \neq (b_n) = t$, then, WLOG, there exists $N \in \mathbb{N}$ such that $a_N = 1$ and $b_N = 0$. Let $q_N : \mathbb{H} \to C_N$ be the retraction that collapses all circles to x_0 except for the N^{th} circle. Then in $\pi_1(C_N) \cong \mathbb{Z}$, $[q_N \circ \ell_N] = [q_N \circ \gamma_s] = 1$, whereas $[q_N \circ \gamma_t] = 0$. So $[\gamma_s] \neq [\gamma_t]$, and the map $s \mapsto \gamma_s$ is injective. □

It seems that we are able to take *infinitary products* inside the Hawaiian group. However, the algebraic definition of a group formally only allows finitary products. For a discussion on what makes this possible in the Hawaiian group, see the last section of this chapter, and for more comprehensive treatment of this phenomena refer to [Bra18, CC00].

7.2 Combinatorial Structure of the Hawaiian Group

In Theorem 6.1, we showed that the Hawaiian earring is homeomorphic to the inverse limit of the system

$$\ldots \xrightarrow{r_{n+1}} X_{n+1} \xrightarrow{r_n} X_n \xrightarrow{r_{n-1}} \ldots \xrightarrow{r_2} X_2 \xrightarrow{r_1} X_1,$$

where $X_n = \bigcup_{k=1}^n C_k$ with the subspace topology from $\mathbb{R}^2$, and for each n, r_n is the retraction collapsing the C_{n+1} to the basepoint. Notice that X_n is homeomorphic to $\bigvee_{k=1}^n \mathbb{S}^1$, and we have computed the fundamental group of the wedge of n circles as $\pi_1\left(\bigvee_{k=1}^n \mathbb{S}^1\right) \cong F_n$ generated by $g_1, g_2, \ldots, g_n$, where g_i is represented by the loop traversing the n^{th} circle. Applying π_1 to the whole inverse system, we get the following inverse system of groups.

$$\ldots \xrightarrow{r_{n+1\#}} F_{n+1} \xrightarrow{r_{n\#}} F_n \xrightarrow{r_{n-1\#}} \ldots \xrightarrow{r_{2\#}} F_2 \xrightarrow{r_{1\#}} F_1,$$

where the bonding map $r_{n_\#}$ "kills" g_n and sends every other generator to itself. Taking the limit of this system we get a group $\varprojlim_n F_n$. [1] Let $q_n : \mathbb{H} \to X_n$ be the map collapsing all but the first n circles to the basepoint x_0. Then, by the universal property of $\varprojlim_n F_n$, the family of maps $q_{n_\#} : \pi_1(\mathbb{H}) \to F_n$ induces a canonical homomorphism $\Psi : \pi_1(\mathbb{H}) \to \varprojlim_n F_n$. It turns out that the map Ψ is injective, which allows us to understand the Hawaiian group as a subgroup of $\varprojlim_n F_n$. This is called the *Shape Injectivity Theorem*, which was first proved by Morgan and Morrison [MM86]. This result is far from trivial and we will devote the next two sections to it, following

[1] This group is known as the first Čech homotopy group or the first shape group of $\mathbb{H}$, and is usually denoted by $\check{\pi}_1(\mathbb{H})$. There is a general construction for Čech homotopy groups, but it is not always defined via inverse limits as it is here.

another proof due to Eda and Kawamura. [EK98]. But before we do this, let us see how this theorem helps us understand the combinatorial structure of the Hawaiian group.

Recall that elements of $\varprojlim_n F_n$ are sequences $(w_1, w_2, \dots)$ of words with $w_n \in F_n$ and $q_{n\#}(w_{n+1}) = w_n$. In other words, removing all instances of the letter g_{n+1} from the word w_{n+1} and then reducing gives us precisely the word w_n. The question then remains: what is the image of Ψ?

Proposition 7.2. *Ψ is not surjective.*

Proof. Consider the sequence $(w_n) \in \varprojlim_n F_n$, where $w_n = (g_1 g_2 g_1^- g_2^-)(g_1 g_3 g_1^- g_3^-) \dots (g_1 g_n g_1^- g_n^-)$. Note that as n goes to infinity, the number of appearances of the letter g_1 grows arbitrarily. Thus, if Ψ were surjective, there would be a loop γ traversing C_1 infinitely often. Assume this is the case and consider the loop $q_1 \circ \gamma : I \to X_1 \cong \mathbb{S}^1$. We can lift it to the universal cover $\mathbb{R}$ of X_1 starting at 0 to get path $\widetilde{q_1 \circ \gamma} : I \to \mathbb{R}$ onto $[0, \infty)$. But this is impossible due to compactness. □

This gives us some idea of what kind of loops should be excluded.

Definition 7.3. *Given $w \in F_n$, define $\#_k(w)$ for $1 \leqslant k \leqslant n$ to be the number of appearances of g_k or g_k^- in the reduced word w. A sequence $(w_n) \in \varprojlim_n F_n$ is called* locally eventually constant *if for every $k \geqslant 1$ there exists $N \in \mathbb{N}$ such that $\#_k(w_n)$ is constant for all $n > N$; i.e. the sequence $(\#_k(n))_{n \geqslant 1}$ is eventually constant for every k. Define $⨳^{\sigma}_{\mathbb{N}} \mathbb{Z} \leqslant \varprojlim_n F_n$, called the free σ-product of $\mathbb{Z}$, to be the subgroup of locally eventually constant sequences.*

Theorem 7.4. *Ψ embeds $\pi_1(\mathbb{H})$ isomorphically onto $⨳^{\sigma}_{\mathbb{N}} \mathbb{Z}$.*

Proof. We will prove injectivity in Theorem 7.14, but for now let us focus on surjectivity. If $\Psi([\gamma]) = (w_n)$ is not locally eventually constant, then there is some k such that the number of appearances of g_k and g_k^- in (w_n) is unbounded. This gives a contradiction as in Proposition 7.2. Therefore, $\operatorname{Im} \Psi \subseteq ⨳^{\sigma}_{\mathbb{N}} \mathbb{Z}$.

Let $\mathcal{C} \subset I$ be the middle-third Cantor set. Let $\bigcup_{k \geqslant 1} (a_k, b_k)$ be its complement $I \backslash \mathcal{C}$, which consists of countably many open intervals. Define a loop $\gamma : I \to \mathbb{H}$ by setting $\gamma(c) = x_0$ for all $c \in \mathcal{C}$, and letting γ on every interval (a_k, b_k) be either constant at x_0 or the loop corresponding to g_{n_k} for some $n_k \geqslant 1$. In that way, we can define a loop corresponding to any sequence $(w_n) \in ⨳^{\sigma}_{\mathbb{N}} \mathbb{Z}$. More formally, this follows from the fact that there is an order preserving injection from any countable totally ordered set (in this case the pseudo-generators in the transfinite word corresponding to (w_n)) into a dense totally ordered set (in this case the set of open intervals in $I \backslash \mathcal{C}$ with the natural order it inherits from I). □

7.3 Trees and Dendrites

Although the Hawaiian earring has no universal covering, each space X_n in the inverse system we described does. Given a free group F_n and a set of generators

$A = \{a_1, \ldots, a_n\}$ for F_n, define its *Cayley graph* $\tilde{X}_n$ as follows: put a vertex for every element of F_n, and if two vertices $g, h \in F_n$ are related by an equation of the form $g = ha_k$ for some $1 \leqslant k \leqslant n$, connect the two vertices by an edge labeled a_k. Define a map $p_n : \tilde{X}_n \to X_n \cong \bigvee_{k=1}^n \mathbb{S}^1$ by sending every vertex to the basepoint x_0 and mapping the interior of every edge with the label a_k homeomorphically to the k^{th} copy of $\mathbb{S}^1 \backslash \{x_0\}$.

Proposition 7.5. *The map $p_n : \tilde{X}_n \to X_n$ is a universal covering.*

Proof. Continuity is immediate from the definition. The preimage of a sufficiently small neighborhood not containing x_0 is a countable union of disjoint intervals homeomorphic to it. Moreover, the preimage of a sufficiently small neighborhood of x_0 (not containing any full circle) is a countable union of neighborhoods homeomorphic to that neighborhood of x_0, where the k^{th} circle comes from the edge labeled a_k.

So it remains to show that $\tilde{X}_n$ is simply connected. Suppose we have a loop in $\tilde{X}_n$, which means that there is a word $w \in F_n$ and generators $a_{k_1}, a_{k_2}, \ldots, a_{k_m}$ such that $w\, a_{k_1} a_{k_2} \ldots a_{k_m} = w$. Multiplying both sides from the left by w^{-1}, we see that there must be a reduction of $a_{k_1} a_{k_2} \ldots a_{k_m}$ to the empty word. But since any instance of the loops corresponding to $a_k a_k^{-1}$ can be contracted to its basepoint, this reduction gives a nullhomotopy on the whole loop. □

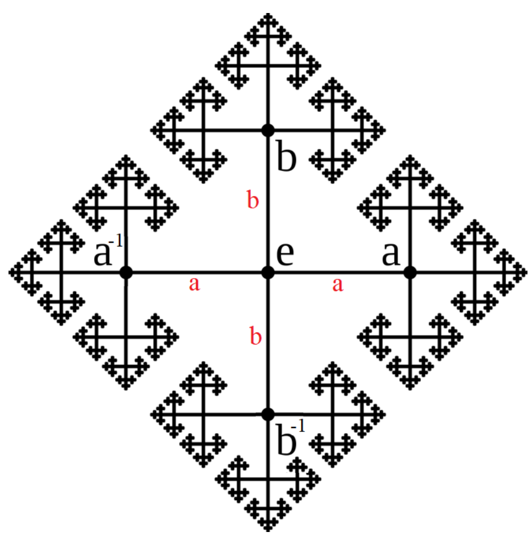


Figure 7.1 Cayley graph of the free group generated by $\{a, b\}$. Edge labels are in red and vertex labels in black

The following is an important property for the study of wild spaces

Definition 7.6. *An* arc *in X is a subspace of X homeomorphic to I. X is called* uniquely arc-wise connected *if for any two distinct points $x, y \in X$, there is a unique arc in X whose endpoints are x and y.*

Recall that a simple closed curve in X is a subspace of X homeomorphic to $\mathbb{S}^1$. We have the following characterization of uniquely arc-wise connected spaces.

Proposition 7.7. *If X is uniquely arc-wise connected, then X is path connected and contains no simple closed curve. The converse holds if X is weakly Hausdorff (i.e. if the image under a continuous function of a compact space in X is closed).*

Proof. Since $\mathbb{S}^1$ is not uniquely arc-wise connected, nor is a space containing a simple closed curve. Assume that X is weakly Hausdorff, path connected, but *not* uniquely arc-wise connected. So we have two arcs A, B with the same endpoints. WLOG, there is a point $a \in A \backslash B$. A and B are closed in X since it is weakly Hausdorff, and so $A \cap B$ is closed in A and $A \backslash (A \cap B) = A \backslash B$ is open in A. Let $u : I \to A$ be a homeomorphism, and let (c, d) be the component of $u^{-1}(A \backslash B)$ containing $u^{-1}(a)$. Let $A' = u([c, d])$ and B' is a subarc of B with endpoint $u(c)$ and $u(d)$, then $A' \cap B' = \{u(c), u(d)\}$ and $A' \cup B'$ is a homeomorphic image of $\mathbb{S}^1$, i.e. a simple closed curve. □

Remark: Containing no simple closed curve and being simply connected are independent conditions. For example $\mathbb{R}^2$ contains simple closed curves and it is simply connected. Conversely, the non-Hausdorff space $X = \{a, b, c, d\}$ with the topology $\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$, called the pseudo-circle, has the same fundamental group as $\mathbb{S}^1$ while containing no simple closed curve!

Proposition 7.5 shows that the universal covers $\tilde{X}_n$ of the approximating spaces for $\mathbb{H}$ are trees. We wish to have a space that retains enough of the properties of universal coverings for $\mathbb{H}$. A reasonable first guess would be to try to form an inverse system of the spaces $\tilde{X}_n$ and consider their inverse limit. It is therefore useful to understand the inverse limits of trees. It is easy to see that trees are contractible and uniquely arc-wise connected. Now given an inverse system of trees

$$\dots \xrightarrow{f_{n+1}} T_{n+1} \xrightarrow{f_n} T_n \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} T_2 \xrightarrow{f_1} T_1,$$

if two points are mapped by f_n to the same point, then the unique arc connecting them is mapped to a subtree of T_n. Therefore, we should not have any simple closed curve after we glue the trees in the inverse limit. Before we prove this, we need the following lemma.

Lemma 7.8. *Suppose A is an inverse limit of a system $\dots \xrightarrow{f_2} A_2 \xrightarrow{f_1} A_1$ of Hausdorff spaces, with projection maps $g_n : A \to A_n$. If X and Y are disjoint compact subsets of A, then there is $n \geqslant 1$ such that $g_n(X) \cap g_n(Y) = \varnothing$. Moreover, $g_k(X) \cap g_k(Y) = \varnothing$ for every $k \geqslant n$.*

Proof. The last statement follows immediately from the preceding one by contraposition. Assume towards a contradiction that for every $n \geqslant 1$ there are $x_n \in X$ and $y_n \in Y$ such that $g_n(x_n) = g_n(y_n)$, and so $g_k(x_n) = g_k(y_n)$ for $1 \leqslant k \leqslant n$. By compactness of X and Y, we can find two subsequences $\{x_{n_j}\}$ and $\{y_{n_j}\}$ that converge to $x \in X$ and $y \in Y$ respectively. Since A is Hausdorff as in inverse limit of Hausdorff spaces, limits of sequences in A are unique; so it suffices to show that $\{y_{n_j}\}$ also converges to x.

Since the inverse limit is a subspace of the product, the neighborhoods of a of the form $U = A \cap \prod_n U_n$, where $U_n \subseteq A_n$ is a neighborhood of $g_n(a)$ and $U_n = A_n$ for all but finitely many n, for a basis form all neighborhoods of a in A. Since $f_n \circ g_{n+1} = g_n$, and by continuity of f_n, we may replace U_{n+1} by $U_{n+1} \cap W_{n+1}$, where $f_n(W_{n+1}) \subseteq U_n$. Repeating this process finitely many times until the maximum n for which $U_n \neq A_n$, we get a basis of neighborhoods of a in A consisting of sets $U = A \cap \prod_n U_n$ such that there exists $N \in \mathbb{N}$ with $f_n(U_{n+1}) \subseteq U_n$ for $1 \leqslant n < N$ and $U_n = A_n$ for all $n > N$.

Let U be an arbitrary neighborhood of x of the type described above. Since $\{x_{n_j}\} \to x$, there exists $J \in \mathbb{N}$ such that for all $n \geqslant 1$, $x_{n_j} \in U_n$ for every $j \geqslant J$. We choose J large enough so that $n_J > N$, and let $j \geqslant J$. Since $g_{n_j}(x_{n_j}) = g_{n_j}(y_{n_j})$, we also get $g_k(x_{n_j}) = g_k(y_{n_j}) \in U_k$ for all $1 \leqslant k \leqslant n_j$. Additionally, $n_j \geqslant n_J > N$ implies that $g_k(y_{n_j}) \in U_k$ for all $1 \leqslant k \leqslant N$ (and also trivially for $k > N$). Therefore, for all $j \geqslant J$, $y_{n_j} \in U$ and $\{y_{n_j}\}$ converges to x. □

Theorem 7.9. *An inverse limit of trees contains no simple closed curves.*

Proof. Let $\ldots \xrightarrow{f_2} T_2 \xrightarrow{f_1} T_1$ be an inverse system of trees and let $X = \varprojlim_n T_n$ be their inverse limit with projection maps $g_n : X \to T_n$. X is Hausdorff as an inverse limit of trees which are themselves Hausdorff. Assume towards a contradiction that $f : \mathbb{S}^1 \to X$ is an embedding of a simple closed curve, and for $i = 1, 2, 3, 4$ let S_i denote the image under f of the intersection of $\mathbb{S}^1$ with each of the four quadrants of the complex plane (including the bounding axes and arranged counterclockwise starting with the positive-real positive-imaginary quadrant). Note that $S_1 \cap S_3 = S_2 \cap S_4 = \varnothing$. Let $x = f(1)$ and $y = f(-1)$. Let u_i be the path traversing S_i in an orientation such that $u_1 \cdot u_2$ and $u_4 \cdot u_3$ are the two arcs in $f(\mathbb{S}^1)$ from x to y.

By the previous lemma, there exists $m \in \mathbb{N}$ such that $g_m(S_1) \cap g_m(S_3) = g_m(S_2) \cap g_m(S_4) = \varnothing$. In particular $g_m(x)$ and $g_m(y)$ are distinct and are thus connected by a unique arc in T_m. Let $\alpha : I \to T_m$ traverse this arc from $g_m(x)$ to $g_m(y)$. So α is the reduced (i.e. containing no nullhomotopic subpath) representative of both $g_m \circ (u_1 \cdot u_2) = (g_m \circ u_1) \cdot (g_m \circ u_2)$ and $g_m \circ (u_4 \cdot u_3) = (g_m \circ u_4) \cdot (g_m \circ u_3)$. Considering the homotopy from $(g_m \circ u_1) \cdot (g_m \circ u_2)$ to α, we see that an initial segment $\alpha\big|_{[0,s]}$ lies in $g_m(S_1)$ and a terminal segment $\alpha\big|_{[s,1]}$ lies in S_2. Similarly, there is $t \in I$ such that $\alpha\big|_{[0,t]}$ lies in $g_m(S_4)$ and $\alpha\big|_{[t,1]}$ lies in S_3. Therefore, one of the following cases occur:

- $s < t$ so that $\alpha([s,t]) \subseteq S_2 \cap S_4$;
- $t < s$ so that $\alpha([t,s]) \subseteq S_1 \cap S_3$; or
- $s = t$ so that $\alpha(s) \in S_1 \cap S_2 \cap S_3 \cap S_4$.

In every case, we get a contradiction. □

Corollary 7.10. *Every path-component of an inverse limit of trees is uniquely arc-wise connected.*

Before we prove shape injectivity, we need to introduce one last tool from continuum theory. First we make the following definitions.

Figure 7.2 the arc hedgehog

Definition 7.11. *A* Peano continuum X *is a Hausdorff space such that there is a surjection* $I \to X$.[2] *A* dendrite *is a Peano continuum containing no simple closed curve.*

A typical example of a dendrite is the so-called arc hedgehog, denoted by $ah(\omega)$, which is the countable wedge of intervals of length $1/2^n$ glued at an endpoint. It is not hard to see that the arc hedgehog can be realized as an inverse limit of trees as shown in Figure 7.2. The following theorem tells us that this is always true for dendrites. For a proof, see Theorem 10.27 of [Nad92].

Theorem 7.12. *Every dendrite is homeomorphic to the inverse limit of trees* T_n, *where* $T_1 = I$, $\overline{T_{n+1} \backslash T_n}$ *is an arc, and the bonding retractions* $r_n : T_{n+1} \to T_n$ *collapse the the arc* $\overline{T_{n+1} \backslash T_n}$ *to the attachment point* $\overline{T_{n+1} \backslash T_n} \cap T_n = \{p_n\}$.

The next theorem will be crucial for the proof of shape injectivity.

Theorem 7.13. *Dendrites are contractible.*

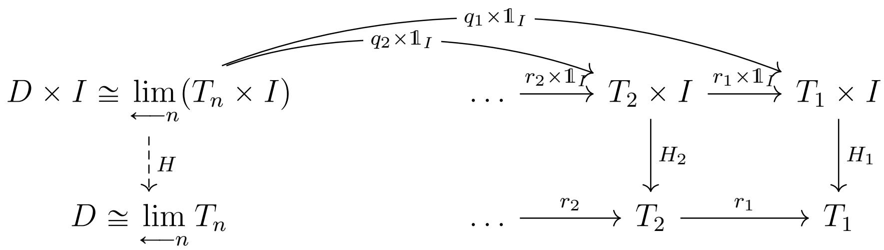
Proof. Suppose D is a dendrite. From the previous theorem, $D = \varprojlim_n T_n$ where $\ldots \xrightarrow{r_2} T_2 \xrightarrow{r_1} T_1 = I$ are trees with each retraction r_n contracting a single arc added. Let $v_0 = 0$ be the basepoint of T_1, and since the bonding maps r_n are retractions, we may take v_0 to be the basepoint of all trees T_n. We wish to contract D to the point $x_0 = (v_0, v_0, \ldots)$. We will proceed by induction. Let $H_1 : T_1 \times I \to T_1$ be the contraction given by $H_1(x,t) = tx$, and suppose that we have constructed a contraction $H_{n-1} : T_{n-1} \times I \to T_{n-1}$ with $H_{n-1}(x,0) = v_0$ and $H_{n-1}(x,1) = x$. Let A be the added arc $\overline{T_n \backslash T_{n-1}}$ and let a_0 be its point of attachment. Let $G_n : T_n \times I \to (T_{n-1} \times I) \cup (A \times \{1\})$ be a retraction satisfying $G_n(A \times I) \subseteq (\{a_0\} \times I) \cup (A \times \{1\})$ and $G_n(A \times \{0\}) = \{(a_0, 0)\}$. Define $K_n : (T_{n-1} \times I) \cup (A \times \{1\}) \to T_n$ on $T_{n-1} \times I$ to be the same as H_{n-1}, and on $A \times \{1\}$ by $K_n(a,1) = a$. Then the map $H_n := K_n \circ G_n$ is a contraction of T_n extending H_{n-1}.

After constructing the contractions in this way, we get the following commutative diagram.

$$\begin{array}{ccccc} \ldots & \xrightarrow{r_2 \times \mathbb{1}_I} & T_2 \times I & \xrightarrow{r_1 \times \mathbb{1}_I} & T_1 \times I \\ & & \downarrow{\scriptstyle H_2} & & \downarrow{\scriptstyle H_1} \\ \ldots & \xrightarrow{r_2} & T_2 & \xrightarrow{r_1} & T_1 \end{array}$$

[2] Peano continua are sometimes defined as connected, locally path-connected, compact, metrizable spaces. The equivalence of these two definition is the statement of Hahn–Mazurkiewicz Theorem (see Theorem 8.18 of [Nad92]).

Since the bonding maps of the top inverse system act as the identity on the interval, we have $\varprojlim_n (T_n \times I) \cong (\varprojlim_n T_n) \times I \cong D \times I$. Let $q_n : D \times I \to T_n \times I$ be the projection maps. Then, we get projections to $H_n \circ q_n : D \times I \to T_n$ to the bottom inverse system, which induce a canonical map $H : D \times I \to \varprojlim_n T_n \cong D$.



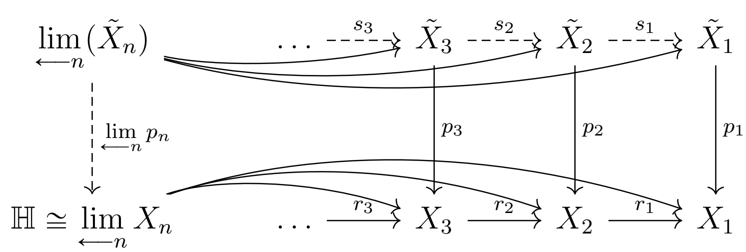
Finally, if $d \in D$, then $H(d, 0) = (H_1(q_1(d), 0), H_2(q_2(d), 0), \dots) = (v_0, v_0, \dots) = x_0$ and $H(d, 1) = (H_1(q_1(d), 1), H_2(q_2(d), 1), \dots) = (q_1(d), q_2(d), \dots) = d$. Therefore, H is the desired contraction. $\square$

7.4 1-Dimensional Shape Injectivity Theorem

We are finally ready to prove the Shape Injectivity Theorem. Recall that we had an inverse system $\dots \xrightarrow{r_2} X_2 \xrightarrow{r_1} X_1$ whose inverse limit is $\mathbb{H}$. The images under π_1 of the projections of $\mathbb{H}$ onto X_n induce a canonical map $\Psi : \pi_1(\mathbb{H}) \to \varprojlim_n \pi_1(X_n) \cong \varprojlim_n F_n$. Moreover, every space X_n has a universal covering map $p_n : \tilde{X}_n \to X_n$, where $\tilde{X}_n$ is the Cayley graph of F_n, which is a tree.

Theorem 7.14. *The canonical map $\Psi : \pi_1(\mathbb{H}, x_0) \to \varprojlim_n F_n$ injective.*

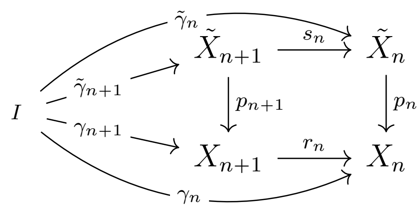
Proof. We will show that the kernel of Ψ is trivial. Suppose $\gamma : I \to \mathbb{H}$ is a loop based at x_0 with $r_n \circ \gamma : I \to X_n$ nullhomotopic for every $n \geqslant 1$. Fix a basepoint $\tilde{x}_n$ in each covering space $\tilde{X}_n$. Since $(r_n \circ p_{n+1})_\# \pi_1(\tilde{X}_{n+1}, \tilde{x}_{n+1}) = \{e\} = p_{n\#} \pi_1(\tilde{X}_n, \tilde{x}_n)$, it follows by Theorem 3.13 that there is a unique lift $s_n : \tilde{X}_{n+1} \to \tilde{X}_n$ of $r_n \circ p_{n+1}$ satisfying $s_n(\tilde{x}_{n+1}) = x_n$. So we get the following diagram.



Moreover the composition of p_n with the projections $\varprojlim_n \tilde{X}_n \to \tilde{X}_n$ induce a map $\varprojlim_n p_n : \varprojlim_n \tilde{X}_n \to \varprojlim_n X_n \cong \mathbb{H}$. Let $\tilde{\mathbb{H}}$ be the path component of $\varprojlim_n X_n$ containing the point $\tilde{x}_0 = (\tilde{x}_1, \tilde{x}_2, \dots)$, and let $p : \tilde{\mathbb{H}} \to \mathbb{H}$ be the restriction of $\varprojlim_n p_n$. The map

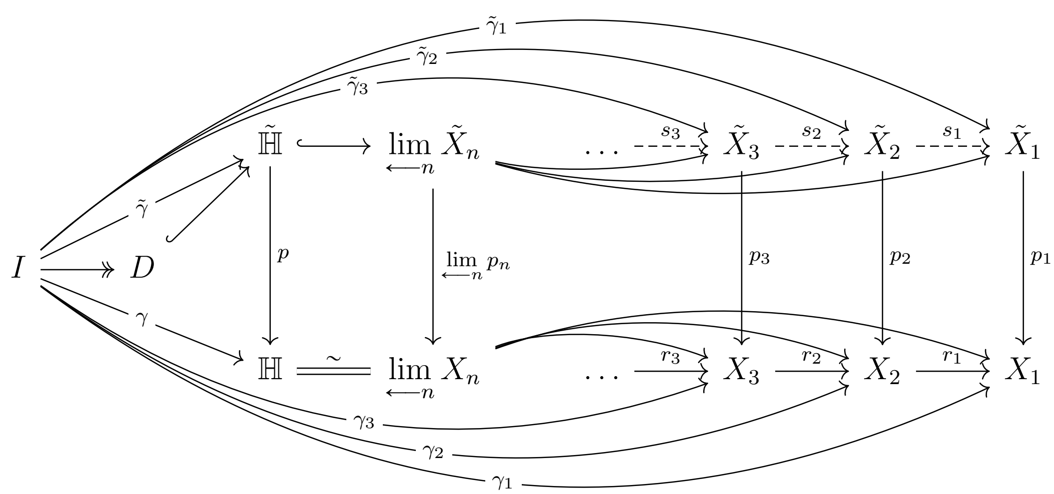
p is not a covering (in fact $\tilde{\mathbb{H}}$ is not even locally path-connected), but it exhibits many of the nice lifting properties of coverings.

By Corollary 7.10, $\tilde{\mathbb{H}}$ is uniquely arc-wise connected. Let γ_n be the projection of γ to X_n, and let $\tilde{\gamma}_n$ be its unique lift to $\tilde{X}_n$ starting at $\tilde{x}_n$ (it will also end at $\tilde{x}_n$ by part (ii) of Theorem 3.9 since γ_n is nullhomotopic). From the following diagram,



$p_n \circ s_n \circ \tilde{\gamma}_{n+1} = r_n \circ p_{n+1} \circ \tilde{\gamma}_{n+1} = r_n \circ \gamma_{n+1} = \gamma_n$. Therefore, $s_n \circ \tilde{\gamma}_{n+1}$ and $\tilde{\gamma}_n$ both lift γ_n and are based at $\tilde{x}_n$, so they are identical by uniqueness of path lifting. Since the loops $\tilde{\gamma}_n$ are compatible with the top inverse system, from the universal property of the inverse limit, we get a canonical loop $\tilde{\gamma} : I \to \varprojlim_n \tilde{X}_n$ based at $\tilde{x}_0$ making the whole diagram commute.

Let $D = \operatorname{Im} \tilde{\gamma}$, which is a Peano continuum by definition. Moreover, since D is path-connected and contains $\tilde{x}_0$, $D \subseteq \tilde{\mathbb{H}}$ and therefore contains no simple closed loop, i.e. it is a dendrite. By Theorem 7.13, D is contractible, so $\tilde{\gamma}$ is nullhomotopic in $\tilde{\mathbb{H}}$ as it factors through a contractible subspace. Therefore, $\gamma = p \circ \tilde{\gamma}$ is nullhomotopic in $\mathbb{H}$ and $\ker \Psi$ is trivial.



□

Notice that throughout the proof, we did not use any specific properties of $\mathbb{H}$, except that it is the inverse limit of spaces whose universal coverings are trees and whose fundamental groups are free. Additionally, the proof of Proposition 7.5 easily generalizes to any graph giving us a tree as a universal covering, and Theorem 4.5 proves that the fundamental group of any graph is free. So we get the following more general version of the theorem.

Theorem 7.15. (1-Dimensional Shape Injectivity Theorem). *Given an inverse system of graphs* $\dots \xrightarrow{r_2} X_2 \xrightarrow{r_1} X_1$, *the canonical map* $\pi_1\left(\varprojlim_n X_n\right) \to \varprojlim_n \pi_1(X_n)$ *into the inverse limit of free groups is injective.*

7.5 Algebraic Wildness of the Hawaiian Group

So far we have seen that there is some notion of multiplication of infinitely many elements of $\pi_1(\mathbb{H})$. In general, however, such an infinitary product is illegal in a group, so there should be some additional structure that $\pi_1(\mathbb{H})$ enjoys, which allow for these infinitary products. We want to instigate this phenomena in a more general setting.

Suppose we have a group G and elements $g_1, g_2, g_3 \dots \in G$. We want to be able to form a product $g_\infty = g_1 g_2 g_3 \dots$. Similarly, we should be able to form any product $g_n g_{n+1} g_{n+2} \dots$ such that

$$g_1^{-1}(g_1 g_2 g_3 \dots) = g_2 g_3 g_4 \dots,$$
$$g_2^{-1} g_1^{-1}(g_1 g_2 g_3 \dots) = g_3 g_4 g_5 \dots, \text{etc.}$$

In other words, we need a sequence of "tails" $t_1, t_2, t_3, \dots \in G$ satisfying $t_1 = g_\infty$ and $t_n = g_n t_{n+1}$ for every $n > 1$. So in order to assign a value in G for the infinitary product, we should find a sequence of tails $\{t_n\}_{n\in\mathbb{N}}$ and check that these relations hold. Still, we run into the following problem. Take an arbitrary $g \in G$ and let $t_1 = g$; we can inductively solve for $t_{n+1} = g_n^{-1} t_n$ so that the infinitary product g_∞ evaluates to g. Since g was arbitrary, our definition for the infinitary product will not be well-defined. Even worse, if we start with any $g \neq e$ and take $g_n = g$ for all n, then $g_\infty = t_1 = ggg \dots = gt_2$; but $t_1 = ggg \dots = t_2$ and so we get $g = e$, a contradiction.

So we need a notion of convergence in order to talk about infinitary products. The following definitions will provide us with such a notion.

Definition 7.16. *A* filtration $\{G_k\}_{k\in\mathbb{N}}$ *in a group* G *is a descending sequence of (not necessarily distinct) subgroups* $G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \dots$. *A sequence* $g_1, g_2, g_3, \dots$ *is said to be* shrinking *in* $(G, \{G_k\}_{k\in\mathbb{N}})$ *if for every* $k \in \mathbb{N}$ *there exists* $N_k \in \mathbb{N}$ *such that for all* $n \geqslant N_k$, $g_n \in G_k$.

Now, if we have a shrinking sequence $\{g_n\}$ and another shrinking sequence of tails $\{t_n\}$ satisfying $t_n = g_n t_{n+1}$ for every n, we will call t_1 *an infinite product of* $\{g_n\}$. The following proposition tells us precisely when this notion is well-defined, i.e. when infinite products of shrinking sequences are unique regardless of the choice of tail sequence.

Proposition 7.17. *Infinite products in* $(G, \{G_k\}_{k\in\mathbb{N}})$ *are well-defined if and only if* $\bigcap_{k\in\mathbb{N}} G_k = \{e\}$.

Proof. If there is a non-identity element $g \in \bigcap_{k\in\mathbb{N}} G_k$, then both of the constant sequences $e, e, \dots$ and $g, g, \dots$ are shrinking and are tail sequences for $e, e, \dots$. So the product $eee\dots$ is defined as both e and g.

To show the converse, suppose infinite products are not unique. Then, we have a shrinking sequence $\{g_n\}$ with two shrinking tails $\{t_n\}$ and $\{s_n\}$, with $s_1 \neq t_1$. Let $g = s_1^{-1}t_1 \neq e$. From the tail condition, we have $s_n s_{n+1}^{-1} = g_n = t_n t_{n+1}^{-1}$, so $s_{n+1}^{-1}t_{n+1} = s_n^{-1}t_n$ for every n. In particular, $s_n^{-1}t_n = s_1^{-1}t_1 = g$ for every n. Given any $k \in \mathbb{N}$, both $\{t_n\}$ and $\{s_n\}$ are eventually in G_k, and since G_k is a subgroup, so is $\{s_n^{-1}t_n\}$, which is the constant sequence $g, g, \dots$. Therefore, $g \in \bigcap_{k\in\mathbb{N}} G_k$ since it is in every G_k. □

We saw that in the case of fundamental groups we can, at least for some spaces, take infinite concatenation of loops by defining the product loop on each of the countable segments $[\frac{n-1}{n}, \frac{n}{n+1}]$. So what is the filtration we have in the case of fundamental groups? Recall that a space is *first countable* at x if there is a countable set of neighborhoods $\{U_k\}_{k\in\mathbb{N}}$ of x such that for any neighborhood U of x there exists $k \in \mathbb{N}$ with $U_k \subseteq U$. Given such a countable collection of neighborhoods $\{U_k\}_{k\in\mathbb{N}}$ around x satisfying $U_k \subseteq U_{k+1}$ for every $k \in \mathbb{N}$, let G_k be the image of the map $\pi_1(U_k, x) \to \pi_1(X, x)$ induced by inclusion. Then, $\{G_k\}_{k\in\mathbb{N}}$ is a filtration on $\pi_1(X, x)$. If X is semi-locally simply connected at x_0, then for all but finitely many $k \in \mathbb{N}$, G_k is trivial; in this case, infinitary products are not as interesting since for a sequence of loops to be shrinking in the filtration, all but finitely many of loops will be nullhomotopic.

Definition 7.18. [Bra20] *A fundamental group $\pi_1(X, x)$ is* infinitary *if there is a loop $\gamma : I \to X$ based x and a closed set $\{0, 1\} \subseteq C \subseteq \gamma^{-1}(x)$ such that $I\backslash C$ has infinitely many components (a_n, b_n), and for every n the subloop $\gamma\big|_{[a_n,b_n]}$ is not nullhomotopic. We call $\pi_1(X, x_0)$ infinitary if it is infinitary at some point $x \in X$; otherwise, it is* finitary*.*

The following characterization of spaces with infinitary fundamental groups illustrates the canonical role the Hawaiian earring plays in wild topology.

Proposition 7.19. *A fundamental group $\pi_1(X, x)$ is infinitary if and only if there is a map $f : \mathbb{H} \to X$ such that $f(x_0) = x$ and the restriction of f to each circle is not nullhomotopic in X.*

Proof. Suppose we have a loop γ satisfying the previous definition, with $\gamma(C) = x$. Then, we have a map $g : I\backslash C \to X$ induced by restricting γ. Since $I\backslash C$ has countable infinite number of components, $\mathbb{H}\backslash\{x_0\} \cong I\backslash C$. Additionally, the restriction of g to each component defines an image of a map on a circle of $\mathbb{H}$ which is not nullhomotopic. So we can extend define the desired function $f : \mathbb{H} \to X$ from g through this isomorphism, and by setting $f(c) = x_0$ for all $c \in C$.

Conversely, suppose we have a map $f : \mathbb{H} \to X$ satisfying the statement of the proposition. Let γ_n be the restriction of f to the n^{th} circle. We can form the infinite concatenation $\gamma = \gamma_1 \cdot \gamma_2 \cdot \dots$ as usual by setting it on $[\frac{n-1}{n}, \frac{n}{n+1}]$ to be a loop homotopic to γ_n and $\gamma(1) = x$. Then, $C = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, 1\right\}$ satisfies the definition. □

Chapter 8

The Fundamental Group of the Harmonic Archipelago

So far we only dealt with fundamental groups of 1-dimensional spaces. In [FZ05], Fisher and Zastrow proved a generalization of the 1-Dimensional Shape Injectivity Theorem for all subsets of $\mathbb{R}^2$. The theorem fails, however, for subsets of $\mathbb{R}^3$. Here we will examine such a space: the harmonic archipelago $\mathbb{HA}$ (defined in Chapter 6), a wild 2-dimensional space that was first introduced in [BS98]. We will give two descriptions of its fundamental group, as a direct limit of a sequence of Hawaiian earring groups and as a quotient of the Hawaiian earring group. The description we give is not the most satisfying, though, as one is left wondering about how its elements look in terms of generators and relations. Indeed, the combinatorial structure of the fundamental group of $\mathbb{HA}$ remains quite mysterious! For more details and properties of $\pi_1(\mathbb{HA})$, see [Fab05, Bra14].

Recall that, in the notation of Chapter 6, $\mathbb{HA} = D\backslash\left(\bigcup_{n\geqslant 1}\mathrm{Int}(E_n)\right)\cup\left(\bigcup_{n\geqslant 1}K_n\right)$, where D is a disk in $\mathbb{R}^2$ on which a copy of $\mathbb{H}$ is drawn such that the basepoint of $\mathbb{H}$ is $x_0=(0,0,0)$, E_n is a disc between the n^{th} circle C_n and the $(n+1)^{th}$ circle C_{n+1} of $\mathbb{H}$, and K_n is a cone of height 1 above E_n. Let $\mathbb{H}_{\geqslant m}=\bigcup_{n\geqslant m}C_n\subseteq\mathbb{H}$. Clearly, $\mathbb{HA}$ is path-connected and locally path-connected. As we discussed earlier, letting a_n denote the top point of K_n, we see that $\lim\limits_{n\to\infty}a_n=(0,0,1)\notin\mathbb{HA}$, so $\mathbb{HA}$ is not compact. Similarly, if X is a compact space and we have a map $f: X\to\mathbb{HA}$, then for any $\varepsilon>0$, $f(X)$ can only hit finitely many of the subsets $T_{n,\varepsilon}=\{(x,y,z)\in K_n \mid z\geqslant\varepsilon\}$ since otherwise we get a sequence $\{b_n\}$ in $f(X)$ converging to $(0,0,z)$ with $z>0$. Therefore, any homotopy $H: I\times I\to\mathbb{HA}$ of the loop ℓ_n traversing the circle C_n can only deform it over finitely many hills by compactness of $I\times I$. So none of the loops ℓ_n is trivial, and yet $[\ell_n]=[\ell_m]$ for all $n,m\in\mathbb{N}$ as the next lemma shows.

Lemma 8.1. *Every loop $\gamma: I\to\mathbb{HA}$ based at x_0 is homotopic to a loop $\gamma_m: I\to\mathbb{H}_{\geqslant m}$ for each $m\geqslant 1$.*

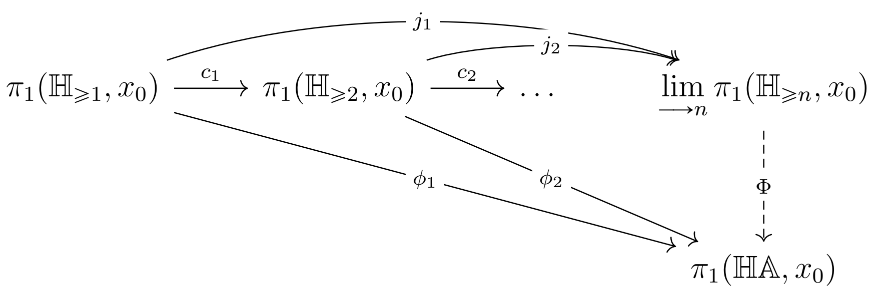
Proof. By the previous remarks, the image of γ lies in a space of the form $Y=\mathbb{HA}\backslash(\bigcup_{n>k}T_{n,\varepsilon})$ for some k. Let D_n denote the subset of Y between C_n and C_{n+1}, which is homeomorphic to a disk, for $1\leqslant n\leqslant k$ and homeomorphic to a punctured disk for $n>k$. Moreover, for every $n>k$ there is a deformation retraction

$F_n : D_n \times I \to D_n$ satisfying $F_n(d, 1) \in C_n \cup C_{n+1}$ for all $d \in D_n$, expanding the hole of D_n into the entire disc. Similarly, there is a deformation retraction flattening the hill K_n into the xy-plane for every $1 \leqslant n \leqslant k$ and then contracting the resulting disk between C_1 and C_{k+1} to C_{k+1}. Combining these two homotopies we get a deformation retraction $F : Y \times I \to Y$ satisfying $F(y, 1) \in \mathbb{H}_{\geqslant k+1}$ for all $y \in Y$. The restriction of F to the image of γ gives us the desired homotopy of loops. □

Corollary 8.2. *The map $\phi : \pi_1(\mathbb{H}, x_0) \to \pi_1(\mathbb{HA}, x_0)$ induced by inclusion is surjective; and if $g_n \in \pi_1(\mathbb{H}, x_0)$ denotes the homotopy class of ℓ_n, then $\phi(g_n) = \phi(g_m)$ for all $n, m \in \mathbb{N}$.*

At this point, one is tempted to think that $\pi_1(\mathbb{HA})$ is quite small; far from it, it is actually uncountable! (see [Fab05] for a proof).

As a direct limit: Let $s_n : \mathbb{H}_{\geqslant n} \to \mathbb{H}_{\geqslant n+1}$ be the retraction mapping C_n homeomorphically onto C_{n+1}, and let $i_n : \mathbb{H}_{\geqslant n+1} \hookrightarrow \mathbb{H}_{\geqslant n}$ denote the inclusion. Since $s_n \circ i_n = \mathbb{1}_{\mathbb{H}_{\geqslant n+1}}$, $(s_n \circ i_n)_\# = s_{n\#} \circ i_{n\#} = \mathbb{1}_{\pi_1(\mathbb{H}_{\geqslant n+1})}$. So $c_n := s_{n\#}$ is surjective and $i_{n\#}$ is injective. We call c_n the *induced retraction on fundamental groups.* Let $\phi_n : \pi_1(\mathbb{H}_{\geqslant n}, x_0) \to \pi_1(\mathbb{HA}, x_0)$ be the homomorphism induced by inclusion. Then, by Corollary 8.2, $\phi_{n+1} \circ c_n = \phi_n$. Therefore, we have a canonical homomorphism $\Phi : \varinjlim_n \pi_1(\mathbb{H}_{\geqslant n}, x_0) \to \pi_1(\mathbb{HA}, x_0)$ by the universal property of the direct limit, as is shown in the diagram,



where j_n denote the maps of the direct limit.

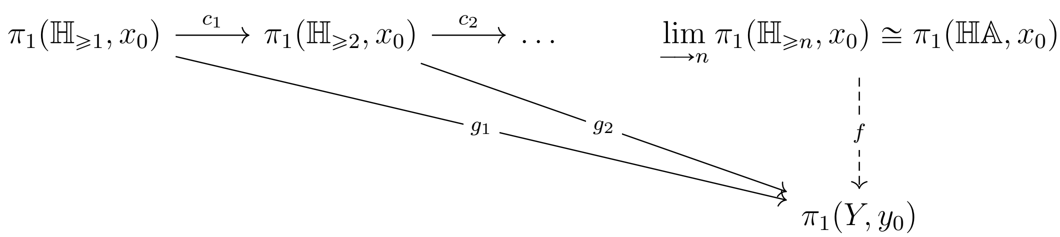
Theorem 8.3. *The canonical map $\Phi : \varinjlim_n \pi_1(\mathbb{H}_{\geqslant n}, x_0) \to \pi_1(\mathbb{HA}, x_0)$ is an isomorphism.*

Proof. By Corollary 8.2, $\phi = \Phi \circ j_1$ is surjective, and thus so is Φ. Since all c_n are surjections, $\varinjlim_n \pi_1(\mathbb{H}_{\geqslant n}, x_0)$ can be identified with a quotient of $\pi_1(\mathbb{H}_{\geqslant 1}, x_0)$. Therefore, to show injectivity, it suffices to prove that for any loop $\gamma : I \to \mathbb{H}_{\geqslant 1}$ with $\phi_1([\gamma])$ nullhomotopic in $\mathbb{HA}$, there exists n such that $c_{n-1} \circ c_{n-2} \circ \cdots \circ c_1([\gamma])$ is the identity in $\mathbb{H}_{\geqslant n}$. If $\phi_1([\gamma])$ is nullhomotopic in $\mathbb{HA}$, then there is a nullhomotopy $H : I \times I \to \mathbb{HA}$ contracting γ to x_0. Again, due to compactness, the image of H only hits finitely many hill tops. Composing with the homotopy F from Lemma 8.1, we obtain a nullhomotopy contracting $s_{n-1} \circ s_{n-2} \circ \cdots \circ s_1 \circ \gamma$ to x_0. Thus, $c_{n-1} \circ c_{n-2} \circ \cdots \circ c_1 \circ \gamma$ is the identity in $\mathbb{H}_{\geqslant n}$, as desired. □

Describing $\pi_1(\mathbb{HA})$ as a direct limit gives us the following universal property.

Corollary 8.4. *Suppose Y is first countable at y_0. For every shrinking sequence of loops $\{\gamma_n\}_{n\geqslant 1}$ based at y_0 such that $\gamma_n \sim \gamma_{n+1}$ for all n, there is a unique induced homomorphism $f : \pi_1(\mathbb{HA}, x_0) \to \pi_1(Y, y_0)$ such that $f([\ell_n]) = [\gamma_n]$.*

Proof. Notice that the condition of the corollary is precisely the same as having a sequence of maps g_n compatible with the direct system of fundamental groups, and the map f is guaranteed by the universal property of the direct limit.



□

As a quotient of $\pi_1(\mathbb{H})$: We wish to describe $\pi_1(\mathbb{HA})$ as a quotient of the Hawaiian group. To do that we will characterize the kernel of the surjective homomorphism $\phi : \pi_1(\mathbb{H}) \to \pi_1(\mathbb{HA})$. Recall that we can represent elements of $\pi_1(\mathbb{H})$ as sequences $(w_n) \in \varprojlim_n F_n$, i.e. where w_n is obtained from w_{n+1} by deleting all instances of g_{n+1}, and the number of times any g_n appears in w_k is eventually constant as $k \to \infty$. Define σ_n on $w_n \in F_n$ as follows: if $n < m$, $\sigma_m(w_n) = 1$, where 1 represents the empty word; otherwise, let $\sigma_m(w_n)$ be the reduced word in $\varprojlim_n$ obtained after replacing each of $g_1, g_2, \dots, g_{m-1}$ by g_m. Define ρ_m on $\varprojlim_n F_n$ by

$$\rho_m(w_1, w_2, \dots) = (\sigma_m(w_1), \sigma_m(w_2), \dots) = (1, 1, \dots, 1, \sigma_m(w_m), \sigma_m(w_{m+1}), \dots).$$

It is clear that for every $m \geqslant 1$, the image of a locally eventually constant sequence under ρ_m is also locally eventually constant.

Proposition 8.5. *If $[\gamma] \in \pi_1(\mathbb{H}, x_0)$ corresponds to $(w_1, w_2, \dots) \in \varprojlim_n F_n$, then $[\gamma] \in \ker\phi$ if and only if there is some $m \geqslant 1$ such that $\rho_m(w_1, w_2, \dots)$ is the identity of $\varprojlim_n F_n$.*

Proof. If $[\gamma] \in \ker\phi$ corresponding to $(w_1, w_2, \dots) \in \varprojlim_n F_n$, then γ is nullhomotopic in $\mathbb{HA}$. Composing this nullhomotopy with the homotopy F from the proof of Lemma 8.1, we obtain a loop γ' which is nullhomotopic in $\mathbb{H}_{\geqslant m}$. Notice that $[\gamma']$ viewed as an element in the image of $\pi_1(\mathbb{H}_{\geqslant m}, x_0)$ under the map induced by the inclusion $\mathbb{H}_{\geqslant m} \hookrightarrow \mathbb{H}_{\geqslant 1}$ corresponds precisely to $\rho_m(w_1, w_2, \dots)$.

For the converse, suppose $\rho_m(w_1, w_2, \dots) = (1, 1, 1, \dots)$ where $(w_1, w_2, \dots)$ corresponds to $[\gamma] \in \pi_1(\mathbb{H}, x_0)$. Then, $c_{m-1} \circ c_{m-2} \circ \dots \circ c_1([\gamma]) = [x_0]$. Therefore,

$$\phi([\gamma]) = \phi_m \circ c_{m-1} \circ c_{m-2} \circ \dots \circ c_1([\gamma]) = [x_0],$$

from the diagram before theorem 8.3. □

Alternatively, we can use van Kampen Theorem to characterize the kernel of ϕ as follow.

Proposition 8.6. *Let* $\phi : \pi_1(\mathbb{H}, x_0) \to \pi_1(\mathbb{HA}, x_0)$ *be the map induced by inclusion. Then,* $\ker \phi$ *is the smallest normal subgroup containing elements of the form* $g_n g_{n+1}^{-1}$ *(called their the conjugate closure).*

Proof. Let $U_0 = \{(x, y, y) \in \mathbb{HA} \mid z < 1/2\}$ be the "chopped" archipelago; and for each $n \geqslant 1$ let B_n be an open disk (to be attached to close the n^{th} hill of U_0, at say height $1/3$), and let R_n be a narrow open ribbon between C_n and C_{n+1} connecting B_n to the basepoint x_0 so that any two such ribbons only intersect at a contractible neighborhood of x_0. Take $U_n = B_n \cup R_n$. Then $\{U_n\}_{n \in \mathbb{N}}$ is an open cover satisfying the hypothesis of van Kampen Theorem. Notice that there is a deformation retraction taking U_0 to $\mathbb{H}$ and so $\pi_1(U_0, x_0) \cong \pi_1(\mathbb{H}, x_0)$. Moreover, $S_n = U_0 \cap U_n$ is a circle, and the generator of its fundamental group is identified by the inclusion $i_n : S_n \hookrightarrow U_0$ with the homotopy class $g_n g_{n+1}^{-1}$. The only nontrivial pairwise intersections in our open cover are of the form $U_0 \cap U_n$, and for each such intersection we have the following diagram.

$$
\begin{array}{ccccc}
 & & \pi_1(U_0, x_0) \cong \pi_1(\mathbb{H}, x_0) & & \\
 & \overset{i_{n\#}}{\nearrow} & & \overset{\phi}{\searrow} & \\
\pi_1(U_0 \cap U_n, x_0) \cong \langle g_n g_{n+1}^{-1} \rangle & & & & \pi_1(\mathbb{HA}, x_0) \\
 & \searrow & & \nearrow & \\
 & & \pi_1(U_n, x_0) \cong \{[x_0]\} & &
\end{array}
$$

By van Kampen Theorem,

$$
\begin{aligned}
\pi_1(\mathbb{HA}, x_0) &\cong \left(\pi_1(U_0, x_0) * (\ast_{n \geqslant 1} \pi_1(U_n, x_0))\right) / N \\
&\cong \left(\pi_1(\mathbb{H}, x_0) * \{[x_0]\}\right) / N \cong \pi_1(\mathbb{H}, x_0)/N,
\end{aligned}
$$

where $N = \ker \phi$ is the smallest normal subgroup containing elements of the form $g_n g_{n+1}^{-1}$ for every $n \geqslant 1$. $\square$

Chapter 9

Further Directions

For spaces that are path-connected and locally path-connected, semi-local simple connectedness is the obstruction to the existence of universal coverings. Understanding the fundamental groups of these wild spaces is algebraically complicated due to the existence of infinitary products, which are not usually encountered in classical combinatorial group theory. For many wild spaces, however, not all hope is lost. We saw that for the Hawaiian earring, there is some sort of generalized covering space, which exhibits some lifting properties that allowed us to identify $\pi_1(\mathbb{H})$ as a subgroup of its so-called first shape group. A theory for such generalized coverings is being systematically developed, for example in [FZ07, Bra15].

As it is shown in [FZ05] building on [CF59, MM86, Eda10], the fundamental groups of any planar set canonically injects into its first shape group. These shape injectivity theorems however fail even for subsets of $\mathbb{R}^3$, for example see [FZ05]. Proposition 7.19 tells us that we can use the Hawaiian earring as a test space for the existence of infinitary products in the fundamental groups of other spaces. Indeed, this was the approach used in Chapter 8 to understand the fundamental group of the harmonic archipelago; still, it was difficult to make sense of its combinatorial structure with this approach. For example, we are unaware of any "word calculus" for the homotopy classes of loops in the harmonic archipelago as we had for the Hawaiian earring.

Understanding the fundamental group of wild spaces in ambient dimension $d \geqslant 3$ remains wide open for research. One way to pursue this may be via examining how they relate to the Hawaiian earring. It is not hard to show from the definitions that the fundamental group of a first countable space is infinitary if and only if it is not semi-locally simply connected at one or more points. Together with Proposition 7.19, this implies that the Hawaiian earring is the natural candidate to investigate the wildness of other spaces. In [KR05, BMM12] for example, the authors study the so-called Hawaiian groups of a space X; the first Hawaiian group of X is defined as $\pi_*(\mathbb{H}, X)$, the group of homotopy classes of basepoint-preserving maps from the Hawaiian earring to X, with the other Hawaiian groups defined similarly using higher-dimensional analogues of $\mathbb{H}$.

The algebraic structure of the fundamental groups of wild spaces is interesting in its own right. There are many questions that have yet to be answered here. What are some other equivalent purely group theoretic criteria for the existence of infinitary products than that of shrinking filtrations (i.e. as in Proposition 7.17)? Can we realize every group with a shrinking filtration as the fundamental group of some wild space? These questions could be the subject of further investigation.

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